De la enseñanza al aprendizaje de las matemáticas

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cesc
Así ha visto Cesc nuestro trabajo.

This is how Cese has seen our work

PREAMBLE

As I finished writing the book, I could not help but feel my hands tremble, a trembling that was linked to the pleasant memories and experiences provided by Dr. Lluís Santaló. The book was finished on 22nd November, 2001, a very significant date for those of us who love education and mathematics, the date on which Dr. Santaló passed away. His influence on education and, in particular, his ties to Catalonia and Latin America have left an indelible mark on the noble profession of education.

Lluís Santaló (1911-2001), an eminent mathematician and educator, was a pioneer in the pedagogical renewal of the teaching of science, in particular mathematics. In 1939 he went into exile in Latin America, where he practised and developed his ideas and methodologies. In 1977 he was awarded an honorary doctorate by the Universitat Politécnica de Catalunya, and in his inaugural speech he expressed his conception of mathematics: “Mathematics is simultaneously art, science and technique. As art, it helps us to discern forms and appreciate nature as a wealth of beauty and harmony; as science, it helps us to know nature and understand its laws; and as technique, it contributes to mastering nature and its forces to put them at the service of citizens.”

PREFACE

An invitation to reflection

Few people, upon reading the title of this work, will have been able to suppress a certain feeling of boredom. Mathematics is a subject that does not usually arouse passion. It is usually considered something as tedious as it is useless. Let us see, if not, what Raymond Smullyan, an American mathematician and author of various popular works, tells us in the Preface of his work The Lady or the Tiger and Other Logic Puzzles, a book about mathematical puzzles and problems. Smullyan tells us that, when he was going to exchange impressions with a child who was reading one of his books, the father, a former classmate, warned him: "He is reading your book and he loves it. But when you talk to him, do not tell him that what he is doing is learning math, because he hates math! If he had the slightest idea that this is actually math, he would stop reading the book immediately."

Many people feel the same way. They associate mathematics with endless hours in front of a blackboard full of formulas, numbers and almost demonic expressions, something truly traumatizing.

It seems clear that our subject is not well regarded in society. Generally, the question “Have you studied mathematics?” is often accompanied by some rather reckless and erroneous reactions:
- People say: “You don’t look it.” Do you have to be a weirdo to get a degree in mathematics? Do you have to look a certain way? Some people think: “Well, he must be very clever.” These people don’t really know what mathematics is; they simply have a mistaken image of it as a discipline that uses a language that is very difficult to understand and for which only a few gifted people are prepared.
- Others limit our science to numbers and counting. This belief is fueled by many primary school teachers who do not foster their students' problem-solving skills and who always use behaviourist methods, doing calculation, calculation and more calculation, and leaving aside the most important part of mathematics, which consists of applying constructivist methods and finding solutions to everyday situations.
- And there are still others who consider that mathematicians are wasting their time, that intellectual efforts "are not necessary", probably because it is much easier to say: "I don't understand", "What good is all this going to do me in life?", "No matter how much you explain it to me, I'm not going to understand", than to make an effort and draw out the intensity needed to overcome the obstacles and thus be able to experience the feeling of pleasure that finally understanding produces.

The outlook for mathematics does not look very encouraging, but appearances can be deceiving. Mathematics is useful and has a future. Its only drawback, if you will, what sets it apart from other studies, is that it is not a sprint, but a long-distance race!

The aim of this text is precisely to demolish the stereotypes that relegate mathematics to the realm of the attainable, the boring and the useless. The aim of this book is to offer an optimistic image of mathematics, and to do so we will focus primarily on two aspects:
a) criticize the standard teaching and learning of mathematics (didactic vision),
b) offer a utilitarian vision of the contents (epistemological vision).

The didactic-epistemology binomial gives us a different vision of mathematics and its role in today's society, and allows us to show it as something close and as one of the most useful tools we have, as something interesting and necessary.

In fact, the aim is to show the friendly side of mathematics, starting from theoretical considerations and providing common, everyday examples for all readers, whatever their professional area and concerns.

It has not been difficult to find these examples because we are immersed in a mathematical soup: product prices, bank interest, credits, materials from the world of construction, elements present in any home, everything has a mathematical interpretation. Do you know that tetrabrick packaging is shaped like a parallelepiped? Do you know any method to remember a missing or forgotten digit in your bank account? We will discuss some of these situations and offer answers, in the most enjoyable and educational way possible, with the aim of serving as educational resources.

The work is structured in two parts. One of them (we could call it "the theoretical framework") invites reflection on the noble profession of education. Starting from the failure of traditional mathematics teaching, both in its methodological aspect and in its content, it advocates other pedagogical trends with a more utilitarian, innovative and realistic orientation.

In the second part (the informative framework) examples are offered of the important formative role of mathematics in various areas of knowledge and of its usefulness as an interdisciplinary subject that integrates various stages of knowledge. These examples can serve as educational resources to enhance the teaching-learning binomial.

Mathematics, what is that?

To begin with, we should ask ourselves a question:
“Do we know what mathematics is?” The word “mathematics” comes from Greek and means “to learn.” The ancient Greeks considered mathematics to be the ultimate knowledge. Today we still see it as something ethereal, only within the reach of a privileged few; however, it is not only the basis of all other knowledge, but it is simpler than much of it.

If we turn to the great thinkers, we cannot forget Descartes. In order to support the statements of the Discourse on Method, Descartes seeks a privileged area of knowledge in which his principles can be clearly exemplified and chooses geometry, together with dioptrics. For him, mathematics constitutes "a way of accustoming the spirit to nourish itself with truths and not to be content with false reasons."

The mathematization of thought is the dogma, sometimes taken to ridiculous extremes, of modern science. Kant goes so far as to say that “in each of the disciplines of nature one can find only as much of true science as there is mathematics in it” (Metaphysical Principles of Natural Science). And the tendency of all areas of knowledge that wish to increase their credibility and prestige has been in recent times, and continues to be, to clothe their methods and objects, in a natural or forced manner, with mathematical structures.

But we must again ask ourselves: Does mathematics deserve this privileged place? There has been no lack of philosophers who have considered this invasive tendency of mathematical thought to be unjustified, and even harmful. Heidegger expresses this question with a play on words: mathematics is "ist nicht strenger, sondern nur enger" ("it is not more exact, but only narrower"). And he is not wrong. Human knowledge contains much more richness than mathematical thought can encompass. There are profound realities that man, more or less consciously, yearns to grasp with knowledge and which escape mathematics. The latter fundamentally dominates the rational component of thought, but it does not even manage to touch on other facets of intellectual knowledge. Between mathematical knowledge and personal knowledge, which involves man in his entirety, there is certainly an abyss. In his *Pensées*, the mathematician and philosopher Blaise Pascal expresses the difference between the mathematical minded (esprit de géométrie) and the intuitively minded (esprit de finesse):

> In the first, the principles are obvious but far removed from common use, so that it is difficult to turn one's head in that direction due to lack of habit. But even if one turns a little, these principles are fully visible, and it would be necessary to have a completely distorted mind to reason badly about principles so crude that it is almost impossible for them to escape us. But in the intuitively minded, the principles are in common use and are before everyone's eyes. One must not turn one's head or be violent. One only needs to have good eyesight; but it is necessary to have really good eyesight, because the principles are so fine and numerous that it is almost impossible for them not to escape us. Now, the omission of a principle leads to error. Therefore, it is necessary to have a clear sight to see them all and also a very healthy mind to not reason falsely about these known principles. [Translation of the Spanish version in the text that is, in turn, a translation of the original French]

The difference between the mathematical and the intuitive mind.—In the one the principles are palpable, but removed from ordinary use; so that for want of habit it is difficult to turn one's mind in that direction: but if one turns it thither ever so little, one sees the principles fully, and one must have a quite inaccurate mind who reasons wrongly from principles so plain that it is almost impossible they should escape notice.

But in the intuitive mind the principles are found in common use, and are before the eyes of everybody. One has only to look, and no effort is necessary; it is only a question of good eyesight, but it must be good, for the principles are so subtle and so numerous, that it is almost impossible but that some escape notice. Now the omission of one principle leads to error; thus one must have very clear sight to see all the principles, and in the next place an accurate mind not to draw false deductions from known principles. [A translation from the French directly into English]

> DIFFÉRENCE ENTRE L’ESPRIT DE GÉOMÉTRIE ET L’ESPRIT DE FINESSE.
>
> En l’un, les principes sont palpables, mais éloignés de l’usage commun; de sorte qu’on a peine à tourner la tête de ce cété-là, manque d'habitude: mais pour peu qu’on s’y tourne, on voit les principes à plein; et il faudrait avoir tout à fait l'esprit faux pour mal raisonner sur des principes si gros qu’il est presque impossible qu’ils échappent.
>
> Mais dans l’esprit de finesse, les principes sont dans l’usage commun et devant les yeux de tout le monde. On n’a que faire de tourner la tête ni de se faire violence. Il n’est question que d’avoir bonne vue, mais il faut l’avoir bonne; car les principes sont si déliés et en si grand nombre, qu’il est presque impossible qu’il n’en échappe. Or, l'omission d’un principe mène à l’erreur: ainsi, il faut avoir la vue bien nette pour voir tous les principes, et ensuite l’esprit juste pour ne pas raisonner faussement sur des principes connus. [An original French edition]

We will now show some characterizations of mathematics provided by classical thinkers in the history of this knowledge. Aristotle called it the "science of quantity"; René Descartes, "the science of order and measure." Charles Proteus Steinmetz referred to it as the most exact science whose operations allow absolute demonstration, with mathematical truths being relative and conditional. [Johann] Carl Frederich Gauss stated that mathematics is the queen of sciences, and arithmetic, the queen of mathematics. Eric Temple Bell defined it as the queen and servant of science. Felix Klein considered it to be the science of what is clear in itself. Henri Poincaré emphasized that mathematics does not study objects, but rather relationships between objects; we can replace some objects with simpler ones as long as the relationship between them does not change. Benjamin Pierce concluded that it is the science that obtains necessary conclusions. Later, David Hilbert contributed to its definition saying that it is a game with very simple rules that leaves meaningless marks on a piece of paper. More recently, Alfred North Whitehead claims that, in its broadest sense, it is the development of all kinds of formal, necessary and deductive reasoning. Bertrand Russell adds his own comment and personal assessment, saying that mathematics can be defined as the subject in which one never knows what one is talking about or whether what one says is true. Finally, Julio Rey Pastor writes that it is the "science of sets" and that from finite sets arises, by abstraction, the concept of number as the foundation of all mathematics.

Each of these definitions highlights one aspect or another of mathematics. Some emphasize the formal, abstract and "pure" aspect; others, its applications and uses.
In this work we will focus on the latter.

I hope that after reading this essay you will gain a better understanding of this science and will come to perceive it as something interesting. The expression “the world is permeated with mathematics” is valid for all periods of humanity because counting and comparing are closely related to everyday activities: thinking, speaking and building.

In short, the work aims to be useful both for trainers and for citizens with a critical and innovative sense.

Joan Gómez i Urgellés
28th November, 2001

I. A JOURNEY THROUGH THE GAPS OF MATHEMATICS TEACHING

> Pi's face was masked, and it was understood that none could behold it and live. But piercing eyes looked out from the mask, inexorable, cold and enigmatic.
>
> *'A Mathematician's Nightmare' Bertrand Russell*

1. Teaching and learning

For centuries and until recently, mathematics teaching has focused on lectures, followed by personal study with supporting texts and individual assessment with exams, a method known as "traditional teaching" or "classroom-centred teaching."

However, recent constructivist theories advocate that it is the student (or apprentice) who builds his or her own cognitive structures, based at each moment on the structures he or she already possesses, while the teacher plays the role of collaborator. These theories are translated into specific techniques in the educational field: learning based on real situations (epistemological component) and group learning (heuristic component).

There are various factors that oppose traditional teaching, among which we can highlight two: the rapid evolution of knowledge and technology and in turn the rapid obsolescence of the same—which reduce the usefulness of the content in favour of the capacity for renewal—and the development of communications and computers that enable new forms of learning and communication or interaction between the teacher and the student.

Teaching strategies, and in particular those in the areas known as "science" must be based on pedagogical principles and serve well-defined objectives. This requires a periodic review of curricular objectives, a redefinition of the concept of subject, and the organization of study plans (both content and teaching methodologies) to keep them up to date with social changes.

This text focuses on teaching innovation. General methodological considerations are presented, emphasizing the need for a change in the teaching of mathematics, both in terms of content and methodologies. Mathematical modelling is presented as a valid alternative, a teaching tool based on the study of real-life situations.

Both the theoretical considerations and the examples presented in the text can be a reference for any teaching professional. They are presented as an invitation to reflect on how to experiment differently with the methodology adopted in teaching centres.

2. General reflections on the teaching of mathematics and its role

It is easy to see the social impact of developments in the various areas of knowledge: companies are introducing devices that, like real robots (programmable automata), replace the artisan worker, computers speed up accounting transactions, even money that is found in the well-known prepayment cards.

However, the teaching of mathematics has not found anyone who can make it evolve. In general, mathematics is taught in the same way as a hundred years ago, in black and white; the only change experienced is the replacement of the traditional trigonometry and logarithmic tables by pocket calculators, perhaps at the urging of the students.
Moreover, teachers often follow the same textbook throughout their teaching career, copying the content on the blackboard word for word. What is more unfortunate is that the content is the same whether the students are future chemists or future economists. In this way, mathematics is kept away from its application to the future professional's speciality.

The problems posed in mathematics class are always the same, the data are always good enough and far from reality so that the results are, as the students say, exact (we will understand "integers"). But this way of operating only serves to disconnect mathematics more and more from everyday situations.

After attending various debates on the orientation of mathematics teaching, we can draw some conclusions:

- Mathematics teachers do not have sufficient knowledge of other areas of knowledge and this leads to a less than utilitarian approach to the content.
- There are teachers who think that it is too late for them, that they are too old to use new technologies and methodologies.
- Many teachers feel comfortable teaching subjects as they were taught, yet suffer from a lack of self-confidence that prevents them from making changes.
- Including data and topics related to everyday life requires continuous work and many teachers prefer to save effort and explain the same thing every year.

Organizing visits and reading newspapers (for example, to explain contour lines from a newspaper clipping of the weather map, to interpret economic graphs, etc.) requires special dedication. There is a certain fear of innovation. Working on new topics also implies proposing evaluations that are consistent with said topics, a fact that breaks the patterns of traditional and habitual evaluation. According to Santaló (1975), "when someone tells a mathematics teacher that he has to teach new topics, he immediately asks for more class hours." On rare occasions, people think about rearranging the content and redefining the methods. Ortega y Gasset, in his work *Misión de la universidad* [Mission of the University] (1930), pointed out:

> One of the evils that has arisen from the confusion between science and university has been to offer chairs, according to the whims of time, to researchers, who are, almost always, terrible professors who feel that teaching is a theft of hours from their laboratory or archive work.[https://mathfiction1.wordpress.com/2024/06/29/the-mathematicians-nightmare-the-vision-of-professor-squarepunt/]

In general, many of the mathematical concepts taught in schools are disconnected from the real world, so that students do not understand the usefulness of mathematics in their studies. All of this leads us to believe that traditional teaching is totally inadequate for students who aspire to apply mathematics —whether they are future economists, biologist or engineers—. I believe that in the training of future professional users of mathematics, the way of presenting the results must be different from the traditional way, which is more directed at understanding abstract concepts; this direction is very convenient for the training of mathematicians, but unnecessary for the training of users of mathematics (who are those who apply it). These aspects invite us to ask the following questions: is it necessary to give another type of orientation to the teaching of mathematics? Is it necessary to introduce applications in the teaching of mathematics and to encourage debate between pure and applied mathematics?

The answers are obvious. Personally, I think that we need to reorganize the content and methodologies to make what is behind the word "mathematics" something useful, enjoyable and fun. We need to make a strong commitment to innovation and new directions. In mathematics we cannot continue mixing 19th century pedagogy with 21st century technology. And as Bob Dylan said in an apologia for innovation, "the times they are a-changin'."

This book is based on the hypothesis that traditional teaching of mathematics is not suitable for training citizens/professionals, although it may be suitable for training future mathematicians, and attempts to offer a new orientation to the formative nature of mathematics.

Figure 1.1 Traditional mathematics teaching "fills students with enthusiasm." [sic] (Photograph published in La Vanguardia, 16th December, 1996.)

As an alternative to teaching, *mathematical modelling* is proposed as a valid and effective teaching/learning tool. Although this chapter briefly presents the idea of modelling, I suggest reading the last chapter of the book, in which this technique is developed more broadly and in more detail.

Mathematical modeling consists (in a simplified way) of formulating a real problem in mathematical terms, solving it and, if possible, interpreting the solution in the terminology of the situation posed.

In short, it is about encouraging creativity and motivating students by showing them real-life applications of mathematics and providing them with resources and everyday examples; it is about teaching them to use the techniques they have learned in an everyday context.

I will briefly outline the objectives that are intended to be achieved through the methodology of mathematical modelling:

a) Get students to adopt a creative attitude.

b) Develop their skills in applying mathematics and motivate them to achieve their academic and professional goals.

c) Train students in modelling techniques.

d) Provide an image of mathematics and its teaching that is different from the traditional one.

e) Help to acquire and understand mathematical techniques and concepts from their applications.

Traditionally, the teaching of mathematics has been carried out through theoretical classes in which a large number of mathematical concepts disconnected from reality were explained, so that the student did not see the epistemology of this beautiful science.

The objectives of this text can be summarized in three points:

1. That the reader, and in particular the student, gets the real reason for the studies undertaken.
2. That the reader and the student discover, from real situations, the usefulness of mathematics.
3. Present mathematics as an applied science, in the sense that it is "queen and at the same time servant of the other areas of science."

Mathematics, which according to the dictionary is “the science of quantity and form,” is present in all human activities. It is the most human of all sciences, even though it often seems as if someone or something is trying to distance it from reality to the point that students (users of mathematics) do not fully understand its presence or application.

This text proposes —as an element of debate and at the same time as an educational resource— an articulation of the content of mathematics from an interdisciplinary and creative perspective, using and discovering mathematical knowledge by posing real problems taken from everyday life. It also implies a substantial change in methodology, which takes on a heuristic aspect and uses mathematical modelling techniques and new resources and evaluative elements. This approach implies a fundamental change in the conception of the role of the teacher, in his or her training profile and in the contextual and emergent preparation of material.

II. EVERYDAY MATHEMATICS

Throughout the day we can observe the presence of mathematics in various facets of our everyday environment. In this chapter we will show situations that accompany us every day and that provide us with elements of reflection on the role of mathematics and its influence on our daily life. We can find mathematics at breakfast, at the market, in the office, at the stationery store, at the bank. The sound of the dreaded alarm clock bids us good morning by displaying numbers; we continue to see numbers on the telephone, the bus line, the timetables... we are really immersed in a numerical dance!

In the exposition I have avoided overloading the text with mathematical formulas, but I have considered it appropriate to add some elements so that those who wish to can delve deeper into the theoretical framework. Thus, people who are not "addicted" to mathematics can skip the formulation.

Let us begin, then, to stroll through the lacunas of mathematical endeavour!

1. At breakfast

If we want to talk about everyday things, the most normal thing to do is to start with breakfast. In particular, we will focus on a product of curious and interesting origin, which is becoming less and less common in our diet, despite being excellent for health. We are referring to honey, although what we will talk about will be its producers, the bees, and the hives. Puig Adam, one of the great pedagogues and mathematicians in the history of our country, proposes in one of his high school books ([Matemáticas 7o curso. Obras de texto de Bachillerato] Mathematics 7th grade. Textbooks for High School [Plan 1938], published in 1951) as an exercise "a technical problem solved by bees":

In order to store honey, bees must build panels of individual cells that form a homogeneous mosaic without wasted spaces. They could achieve this goal with triangular, square and hexagonal cells, which are the basic shapes that can be used to form a mosaic. However, in building the entire panel they must also use the least amount of wax possible to achieve the greatest capacity in each cell.

To simplify the exercise, we will reduce the three-dimensional cells to two-dimensional figures, replacing the volume of the cells (which tells us the amount of honey they can store) by the surface area of the figure and the surface area of the cell (which indicates the amount of wax needed to build it) by the perimeter of the figure.

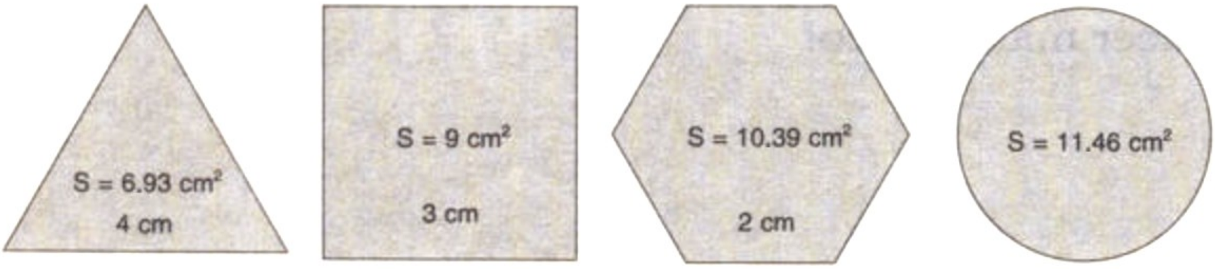
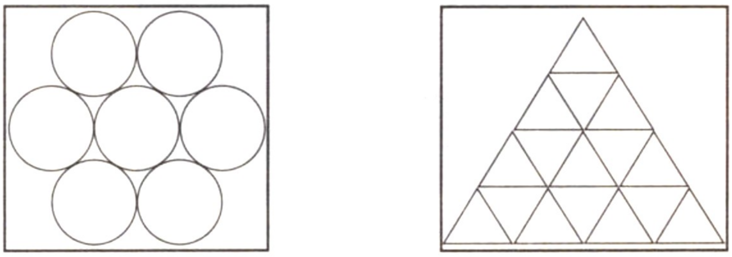


Figure 2.1 Possible honeycomb shapes

Let's see what the surfaces of a triangle, a square, a hexagon and a circle are, all with the same perimeter (for example, 12 cm).

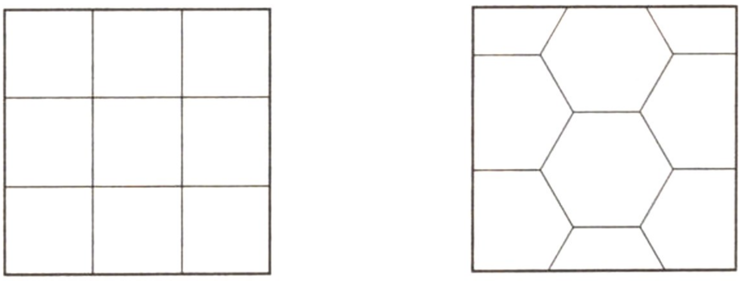
We see that the option with the greatest surface area is the circle; therefore, if we had to build a single-cell panel, we would choose the circular shape because we would obtain the greatest capacity with the least amount of wax used. But when we think of a panel with several cells, taking into account that we have to economize on the total space, the panel with circular cells would be discarded because it would leave useless gaps between the cells. In addition, a mosaic cannot be formed with circles, since there are no common walls between two cells that would allow saving material. Therefore, the shape that ensures the least amount of wax used, the greatest surface area in each cell and, in general, the use of all the resources of material and space is the hexagon. And this is precisely the technique that bees use.

This is a demonstrative example of how in nature, with the help of evolution, the most incredible cases of coincidence between life, efficiency and optimization of resources occur. This informative resource shows an example of geometric optimization in nature.



A mosaic of circles

A mosaic of triangles



A mosaic of squares

A mosaic of hexagons

Figure 2.2 Different types of mosaics

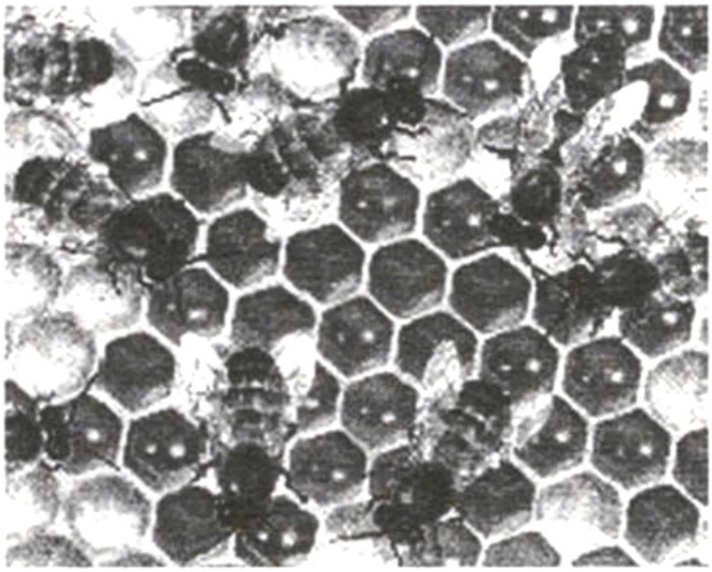
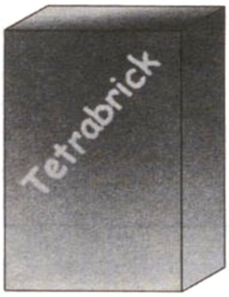


Figure 2.3 Honeycomb

1.1 Milk and its packaging



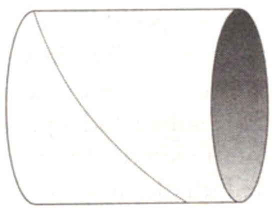
Tetra-pack packaging

It is difficult to find a home where breakfast is not served with a glass of milk or juice from a tetra-brick container. Although glass milk bottles still exist, they are increasingly being replaced by plastic ones; in supermarkets, the vast majority of products are presented in the aforementioned tetra-brick or tetra-pack.

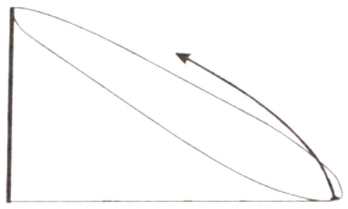
But do we know what this word means and where it comes from? Mathematics can help us clarify the name of this quaint geometric container.

The word is formed from the Greek prefix tetra, meaning "four", and brick, which in English means "brick", or pack, which, also in English, means "package" [?]. We know from experience that these containers are shaped like a parallelepiped with a rectangular base, that is, they do indeed resemble a brick; so we have already justified part of the name. But where does the name tetra come from? Obviously not from the number of sides, since a brick has six sides. No, the explanation is different.

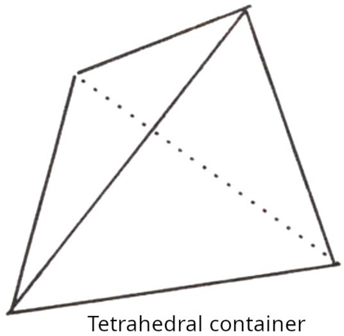
Until a few years ago, all liquids were sold in glass bottles. This type of packaging is now called “ecological” because it is recyclable. However, in those years this aspect was not valued. Since glass bottles were expensive to manufacture, they had to be collected to be reused, that is, they had to go back in time to the one they had taken until then, from the consumer to the bottler, which was a great nuisance and required a great deal of planning. An easy-to-make and disposable container had to be found. However, replacing glass containers would not be an easy task. After many inventions with more or less success, the following solution emerged: a cylinder of aluminium foil (similar to the current one) but made of aluminium sheets and cardboard was considered. The cylinder was closed at one end and an angle cut was made from one corner of that end. The tab thus formed was joined to the origin and the two edges created were welded together. The result was... a tetrahedron, which, as we all know, has four sides; hence the word tetra.



Initial cylinder



Inclined cut



Tetrahedral container

Figure 2.4 Phases in the construction of a tetra-pack

They soon realised, however, that this packaging presented new problems. It was difficult to transport, as it could not be stored in any type of standardised box and its stacking left useless spaces between the containers. For this reason, and despite its initial success, especially for dairy products, it was quickly replaced by the brick-shaped packaging we know today, which was given the name tetra brick. The tetra brick is easily stackable and can be stored in standardised boxes. This example shows us, once again, the role of geometry in solving everyday problems.

2. In a jewellery store window

If you have ever stopped in front of a jewelry store to appreciate the beauty of a piece and to look, with caution and concealed frustration, at its price, you will have noticed that there is a vocabulary specific to these establishments. This vocabulary is used almost exclusively for certain metals and precious stones. Thus, for example, we speak of carats, hundredths, white gold, etc. And you may have asked yourself: "But isn't gold golden in color?" Now we will see what these units of measurement mean and some of the terms used.

2.1. Gold

The term "karat" (K) is used for gold, which, as we will see later, can have several meanings. In this case, it refers to the purity of the gold used in the jewel in question. Thus, for example, we could have 200 grams of 24-carat [the purest] gold.

We start from the assumption that pure gold is 24 carats. If other metals are added to this gold, the alloy that is formed, logically, loses purity and takes on a carat value proportional to the weight of gold remaining. Thus, for example, if we add 25 grams of another metal to 75 grams of pure gold, the result will be 18 carat gold (75% gold and 25% of the other metal). Thousandths are also used, which indicate how many parts of pure gold there are in a thousand parts of the sample. Taking these assessments into account, in the jewelry store window we could distinguish:

- 24-carat or 1,000 thousandths gold: only sold in bars with engravings on the surface for collectors, for investors who buy gold instead of shares and for jewelry workshops. Jewelry is not usually made with 24-carat gold, as it is a very soft material that is not easy to work with.
- 18-carat or 750 thousandths gold, also called "first law" in Spain: this is the one that is usually used in the jewelry field and is what we usually refer to when we talk about gold jewelry, especially in Europe.
- 14-carat gold, 584 thousandths or "second law" in Spain.
- 9 carat or 375 thousandths gold: only commonly used in Great Britain.

When we buy a piece of jewellery, we are told its purity in carats and its weight, but it is not customary to say which metals the alloy is made of. However, there are certain gold alloys that receive a characteristic name. Thus, we have:

- Yellow gold: gold with silver and copper. It is easy to work with.
- Green gold: gold with silver.
- Red gold: gold with copper.
- Rose gold: two parts copper and one part silver.
- Blue gold: 75% gold and 25% iron.
- Palladium white gold: gold with palladium.
- Nickel white gold: gold with nickel.

2.2. Silver

To measure the purity of silver, thousandths are used. We must take into account that silver is very soft, so it is usually alloyed with some other metal, usually copper.

What is known as "sterling silver" is silver of 925 thousandths. That is to say, in sterling silver, of every 1,000 grams of metal, 925 are silver, and the rest, 75 grams, are of another metal. We also have silver of 800 thousandths or "second law silver", composed of 800 grams of silver and 200 grams of another metal.

2.3. Precious stones

Carats also appear here, but this time they mean weight; in this case the exact name is "metric carat". Thus, a metric carat, according to the International Bureau of Weights and Measures, is 200 milligrams.

Diamond weight can also be measured in points, with 100 points equaling 1 carat. Note that we are talking about a 20-carat stone and not 20 carats of stones, meaning that the carat is used to determine the weight of a single stone.

2.4. Pearls

Pearls are measured in pearl grains: 1 grain equals 250 grams.

3. Passing by the stationery store

Every reader will have used sheets of paper to take notes. There are many types and sizes of so-called "desk paper": cuartillas, holandesas, folios, etc. In this chaos of measurements, an attempt was made to bring some order by creating some standards that were included in a generic regulation called DIN [Deutsches Institut für Normung]. The DIN regulation serves to standardize all kinds of objects: connectors, graphics, procedures, etc.

[A cuartilla is a sheet of paper traditionally measuring 157.5 × 215 mm (half a folio). It was called “cuartilla” because it was one-fourth the size of a full sheet of paper. Nowadays, the term “cuartilla” is used to refer to the A5 paper format, which is approximately 148 × 210 mm and corresponds roughly to the old cuartilla size.

Holandesas: In some English-speaking countries, the term “quarto” (not cuartilla) refers to a book size approximately 10 inches (250 mm) in height by 8 inches (200 mm) in width. This size is larger than the cuartilla and is similar to what was known as “holandesa” in the Iberian Peninsula. The name “quarto” comes from it being one-quarter the size of a sheet of paper, approximately equivalent to a double folio.

Folios: A folio is a standard-sized sheet of paper, and it’s twice the size of a cuartilla. It’s commonly used for printing and writing.]

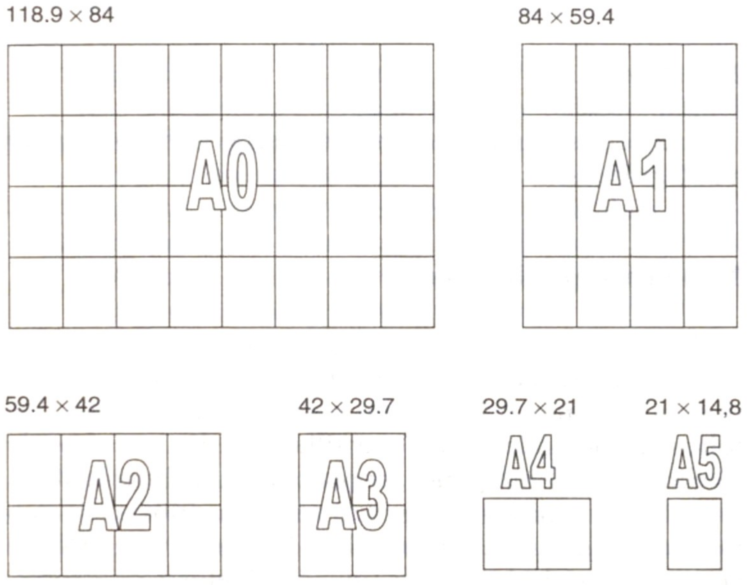
DIN standards were also created for paper measurements. One of the most commonly used measurements is DIN A4, which has strange dimensions of 29.7 cm x 21 cm.

The following table shows the sizes covered by this standard:

Name	Measurements (cm × cm)
A0	118.9 × 84
A1	84 × 59.4
A2	59.4 × 42
A3	42 × 29.7
A4	29.7 × 21
A5	21 × 14.8

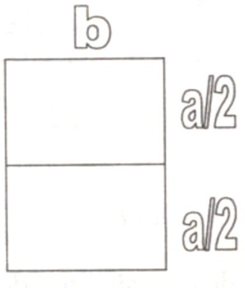
Table 2.1. DIN dimensions

We see that the quotient between the longer side and the shorter side is in all cases 1.41 (the square root of 2).



The origin of this rule is found in the search for a rectangle such that the juxtaposition of two of them produces another rectangle similar (proportional) to the initial one, although larger. So, from the largest to the smallest, all sizes are obtained by dividing the largest into two equal rectangles whose longest side is equal to the smallest of the previous one. With this, it is achieved that, whatever the size needed, the proportions of the same are maintained. And the question is: for this purpose, what measurements are given to rectangles? We will try to clarify where these strange measurements come from and how they are calculated.

Let's start with the larger rectangle. Let's take the lengths "a", for the longer side, and "b", for the shorter side:



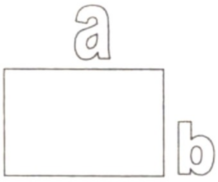
We obtain two rectangles such that, in order for them to be proportional (this criterion is used for the construction of any DIN), the ratio between their sides must be the same; that is: the length of the longer side of the "large" rectangle divided by the length of the shorter side of the same must coincide with the quotient between the longer side and the shorter side of the "small" rectangle. Formally, this gives us the following relationship:

$$\frac{b}{a}=\frac{\frac{a}{2}}{b}\Rightarrow\frac{b}{a}=\frac{a}{2b}\Rightarrow 2b^2=a^2$$

From which it follows that: $a/b=\sqrt{2}$, that is, the quotient between the sides of any DIN must be precisely the square root of 2.

Taking into account the above, we can now justify the construction of a DIN A0.

Another condition was imposed for the construction of the DIN A0, and that was that the largest format, the DIN A0, had a surface area of 1 m^2. This led to the formulation of two equations; calling the sides "a" and "b" gives:



$$\frac{a}{b}=\sqrt{2}$$
$$ab=1$$

The solutions to this system of equations are: *b* = 0.840 m, *a* = 1.189 m.

Therefore, the DIN A0 will measure 118.9 × 84 cm^2. For the rest of the sizes, as shown in the diagrams above, it is enough to divide this size by 2.

That is, the relationship of sheet sizes is as follows:

2A4 = A3
2A3 = A2
2A2 = A1
2A1 = A0

As an application of DIN measurements, we will reflect on their influence on the making of photocopies; to do so, we will first illustrate a result of elementary geometry postulated by Euclid in his text The Elements, from approximately 300 BC [and paraphrased here]:

> If two figures are proportional and "to scale", then if the ratio of their lengths is a value a, then the ratio of their surfaces is a^2 and that of their volumes is a^3; a is called the "proportionality factor".

The classic carbon paper for photocopying is now a thing of the past. Today there is a size, DIN A4, which with the strange measurements mentioned above reproduces any type of document, and can be enlarged or reduced. As we have indicated, we know that the measurements of a DIN A3 are twice that of a DIN A4; this type of paper is the one used in all photocopy shops.

Let us apply Euclid's result:
(*Area DIN A4/area DIN A3*) = (*area DIN A4/2area DIN A4*) = 0.5 = a^2; that is:
a = the square root of 0.5 = 0.707.

In summary: *length of an object in DIN A4/length of an object in DIN A3* = 0.707. In other words:
The *length of an object in DIN A4* is the *length of an object in DIN A3* multiplied by 0.707.

Imagine a graphic on a DIN A3 sheet of 10 cm length; when reduced to DIN A4 its dimensions will be 10 × 0.707. In other words: when reduced, the surface is reduced by 50%, but the object reduces its linear dimensions by 30%, not by half, as some might think!!!

This example is a clear demonstration of the influence that mathematics has on us as critical citizens.

4. Coding and identification

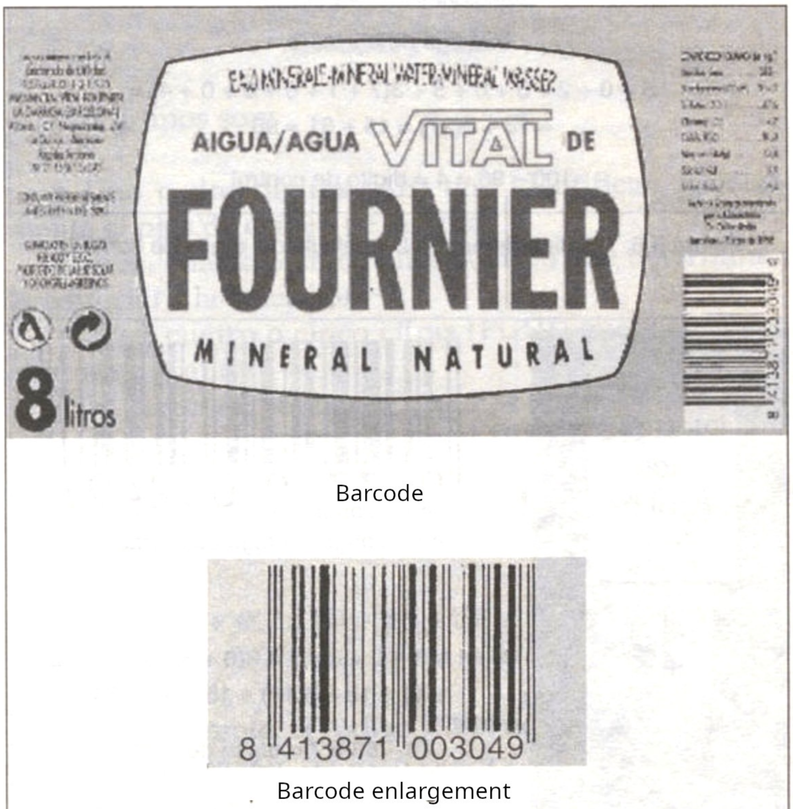
We use bar codes, credit cards, bank accounts every day... Perhaps we have ever needed to remember a number—for example, the so-called "check digit" of a current account. We will see how mathematics can provide us with some help and familiarize us with these collections of numbers that are already part of our society and that coexist with us.

4.1. Barcodes

Barcodes typically consist of thirteen digits, represented by black bars and white spaces, forming a binary code that can be easily read by optical systems.

Each of these thirteen digits (ABCDEFGHIJKLM) represents something different:

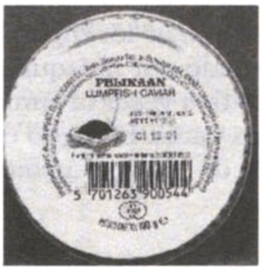
- The first two numbers (AB) are the country of origin code for the product. For example, if the barcode starts with 84, it means that the product comes from Spain, and if it starts with 83, it means that it comes from France.
- The next five (CDEFG) identify the producing company.
- The next five (HIJKL) indicate the product code, assigned by the company.
- The last number (M) is the check digit. To calculate it, we must add the digits in odd positions (starting from the left and not counting the check digit) and add three times the sum of the digits in even positions. The check digit is what is missing from the result of the previous operations from the next multiple of 10.



$$8 + 1 + 8 + 1 + 0 + 0 + 3(4 + 3 + 7 + 0 + 3 + 4) =$$
$$= 18 + 3(21) = 18 + 63 = 81$$

90 – 81 = 9 = control digit

Figure 2.5. A barcode and check digit calculation [#1]



Barcode #2

$$5 + 0 + 2 + 3 + 0 + 5 + 3(7 + 1 + 6 + 9 + 0 + 4) =$$
$$= 15 + 3(27) = 15 + 81 = 96$$

100 – 96 = 4 = control digit

Figure 2.6. A barcode and check digit calculation [#2]

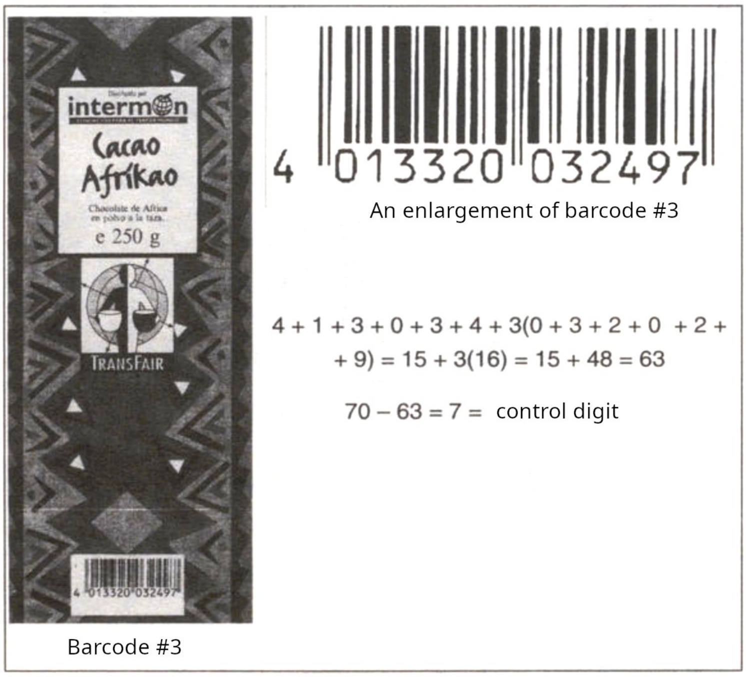


Figure 2.7. A barcode and check digit calculation [#3]

4.2. ISBN code

For the identification of book collections, F. G. Forster introduced the ISBN (International Standard Book Numbers) system in 1964. This system is useful for cataloging and identifying published books.

The ISBN code consists of ten digits divided into four groups, plus the check digit:

AB-CDE-FGHI-J

The groups are:

- The first and second digits (AB) generally identify the country of origin of the work.
- Third, fourth and fifth figures (CDE) determine the publisher and the collection.
- Sixth, seventh, eighth and ninth digits (FGHI) identify the title and the author.
- The check digit (J), which is a number between 0 and 9, or the letter X (which represents 10).

The check digit will be the one that enables the following formula to be completed:

$10A + 9B + 8C + 7D + 6E + 5F + 4G + 3H + 21 + 1J = R$ (R must be a multiple of 11).

For example, the ISBN code of the work L'altra cara de les matemátiques, by Joan Gómez i Urgellés (Vilanova i la Geltrú, El Cep i La Nansa editions, 2000), is 84-85960-45-9. To know if 9 is correct, we will calculate:

$(10 \times 8) + (9 \times 4) + (8 \times 8) + (7 \times 5) + (6 \times 9) + (5 \times 6) + (4 \times 0) + (3 \times 4) + (2 \times 5) + (1 \times 9) = 330$

Since 330 is a multiple of 11, the check digit is correct.

4.3. Credit cards

Credit and debit cards offered by banks and large retailers are essentially made up of the same groups of digits and the same algorithm for calculating the check digit as bar codes and the ISBN code. Few cards have 13, 14, 15 or 17 digits; most have 16 digits, each with a value between 0 and 9, and grouped into groups of four digits for easier viewing:

ABCD EFGH IJKL MNOP

The meaning of each group of digits is as follows:

- The first four digits (ABCD) identify the bank or entity that issues the card; each bank has a different number that may vary depending on the continent. In addition, these four digits are also related to the type of card.

Table 2.2. Bank codes on credit cards

ABCD	Entity	ABCD	Entity
Visa		**Visa**	
4548	La Caixa	4232	Chase Lincoln First Classic
4940	Citibank	4241	Nat. Westminster Bank
4916	Ccat (7electron)	4250	First Chicago Bank
4160	La Caixa (8electron)	4271	Citibank Preferred
4024	Bank of America	4302	H.H.B.C.
4052	First Cincinatti	4310	Imperial Savings
4060	Navy Federal Credit Union	4317	Gold Dome
4128	Citibank (EE UU)	4387	Bank One
4131	State Street Bank	4428	Bank of Hoven
4215	Marine Midland	4019	Bank of America
4225	Chase Manhattan	4811	Bank of Hawaii
4231	Chase Lincoln First Classic	4897	Village Bank of Cincinatti
Mastercard		**Mastercard**	
5215	Marine Midland	5411	1st Fin. Bank of Omaha
5217	Manufacturers Hanover Trust	5414	Nat. Westminster Bank
5233	Huntington Bank	5415	Colonial National Bank
5242	Chevy Chase Federal Savings	5424	Citibank
5254	Bank of America	5465	Chase Manhattan
5263	Chemical Bank	5678	Marine Midland
5273	Bank of America	1263	Chemical Bank
5286	Chase Lincoln First	6207	Marine Midland
5317	Norwest	1033	Manufacturers Hanover Trust
5323	Bank of New York	1226	Huntington
5329	Maryland Bank NA (MBNA)	6017	MBNA
5410	Citibank Preferred	1665	Chase Manhattan

- The fifth digit (E) corresponds to the type of card and indicates which financial institution manages it. As you can see, it does not follow a rigid rule.

Table 2.3. Credit card prefixes

Type	Entity
3	American Express
4, 0, 2	Mastercard
5, 0	Visa
6	Discover

- The next ten digits (FGH IJKL MNO) uniquely identify the user. This identification not only indicates the customer's serial number, but is also related to the card's credit limit and its expiry date (which does not mean that these two pieces of information are clearly indicated; only the group to which the card belongs is indicated). This presents a certain ambiguity, since, as is known, the card's credit limit can vary at the user's request or at the bank's initiative, but this will not cause the card to change group (for example, a Visa Classic will never have the limit of a Visa Gold).
- The final digit is the check digit (P), which is calculated as follows:
  a) Consider the fifteen digits on the card (sixteen minus the control digit).
  b) Each digit in an odd position, starting from the left, is multiplied by two. If the result is greater than 9, the two digits of the number obtained are added together.
  c) The numbers thus calculated are added together and the result is added to the digits located in even positions (including the control digit).
  d) If the result is a multiple of 10, the card numbering is correct. The check digit assigned to a card is the one that makes this total sum a multiple of 10.

Take, for example, the following credit card number:

a) 1234 5678 9012 3452 Let's check if your check digit is correct:
b) 1×2 = 2
3×2 = 6
5×2 = 10; 1 +0 = 1
7×2 = 14; 1+4 = 5
9×2 = 18; 1 + 8 = 9
1×2 = 2
3×2 = 6
5×2 = 10; 1 + 0= 1
c) 2 + 6 + 1+ 5 + 9 + 2 + 6 + 1= 32
2 + 4 + 6 + 8 + 0 + 2 + 4 + 2=28
32 + 28 = 60
d) The result is 60, therefore 1234 5678 9012 3452 is a correct card code.

The mathematical foundation of these algorithms is based on the properties of integers, specifically on what is known in mathematics as "the ring of integers modulo 10", and is useful for these sixteen-digit cards as well as for those with thirteen, fourteen, fifteen and seventeen digits.

4.4. Current accounts (in Spain)

Current accounts are made up of four groups of digits, whose value can vary between 0 and 9:
ABCD EFGH IJ KLMNOPQRST

- The first four digits (ABCD) indicate the bank to which the account belongs.
- The next four (EFGH) identify the office where the account was opened.
- IJ represent the control digits. I is obtained from the bank code (ABCD) and the branch code (EFGH) (for this reason, the first control digit of a branch will always be the same), and J is calculated from the user number (KLMNOPQRST).
- KLMNOPQRST identifies the user. The assignment of this number depends on each entity.

Check digits are calculated as follows:

- The number is considered to be formed by the digits of the banking entity followed by those of the office, adding as many zeros as necessary to obtain a ten-digit number.
- Another number is formed from the number that identifies the user, adding the necessary zeros in front until obtaining ten digits.
- With this we get two ten-digit numbers:
  abcdefghij
  klmnopqrst
- Using the digits of each of the previous numbers, N is calculated in the formula:

$$N = 11 - [(a \times 1) + (b \times 2) + (c \times 4) + (d \times 8) + (e \times 5) + (f \times 10) + (g \times 9) \times (h \times 7) + (i \times 3) + (j \times 6)] \bmod 11$$

'mod' is the modulus function, that is, the remainder given by dividing $[(a \times 1) + (b \times 2) + (c \times 4) + (d \times 8) + (e \times 5) + (f \times 10) + (g \times 9) + (h \times 7) + (i \times 3) + (j \times 6)]$ by 11. That is, if $N = 11$, then N is set to 0; if $N = 10$, then N is set to 1. If N is any other value, then it is left as is. [NB: this definition is a verbatim translation and is different from the standard form of the modulo function where the value 10 would be left unaltered. And is also different from the standard definition of modulus which is usually the unsigned value of a number.]

Calculating N for *abcdefghij* and for *klmnopqrst* we will obtain two values: I and J, the control codes.

Let's see an explanatory example:

If we have the current account number 1234 5678 987654321 and we want to obtain its corresponding control code, we will follow these steps:

- join the bank code and the branch code, adding two zeros in front:
  0012345678 (*abcdefghij*)
- Apply the formula:

$$11 - [(a \times 1) + (b \times 2) + (c \times 4) + (d \times 8) + (e \times 5) + (f \times 10) + (g \times 9) + (h \times 7) + (i \times 3) + (j \times 6)] \bmod 11 = N$$

$$N = 11 - [(0\times1)+(0\times2)+(1\times4)+(2\times8)+(3\times5)+(4\times10)+(5\times9)+(6\times7)+(7\times3)+(8\times6)] \bmod 11$$
$$= 11- 231 \bmod 11 = 11-0=11$$

Since N = 11, it is replaced by 0; N = 0

- Put zeros in front of the current account code until you get ten digits:
  0987654321 (*klmnopqrst*)
- Apply the formula:

$$11 - [(k \times 1) + (l \times 2) + (m \times 4) + (n \times 8) + (o \times 5) + (p \times 10) + (q \times 9) + (r \times 7) + (s \times 3) + (t \times 6)] \bmod 11 = N$$

11 - [(0×1) + (9×2) + (8×4) + (7×8) + (6×5) + (5×10) + (4×9) + (3×7) + (2×3) + (1×6)] mod 11
=11-[255] mod 11 = 11-2 = 9

- Join the two *N* values obtained to form IJ:
  IJ = 09

These more sophisticated algorithms are another example of the power of mathematics and its presence in everyday life, and serve as an educational resource for the training of our students and at the same time for obtaining the respective mathematical models.

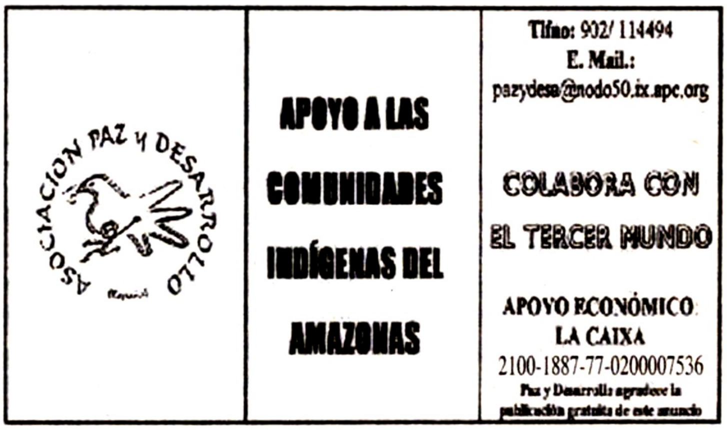


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Figure 2.8. Sample bank account number.

5. Immersed in the units of measurement

In our work life as well as in our leisure life, we are immersed in units of measurement: from grams of sugar to megabytes on our computer. Therefore, it is common sense to want to have a minimum knowledge of units of measurement, their influence, their meaning and their usefulness. We will also see some already obsolete units that are still present in the educational system, which corroborates the thesis that it is necessary to clean up the system.

5.1. Diversity of units

Sometimes we use certain elements around us in such a habitual way that we think that everything is done in the same way everywhere; however, the reality is quite different. The metre, the kilometre, the litre and the kilogram are not, by any means, universal measures, despite all the efforts to introduce them as common units throughout the world. In many territories there are widely used measures, different from the above, which we will discuss later. This leads, even today, to painful confusion and misunderstandings.

Let us suppose that we go to the cinema to see the film Armageddon, which tells the story of the efforts to prevent a huge meteorite from hitting the Earth. At one point in the film a voice says:

“When the meteorite hits the Earth, five billion people will die.” Those who have an idea of the development of the world population, now estimated at 8,100,000,000 (over eight billion) people, will know that if we added up all the people who have lived on this earth throughout the history of mankind, we would not get more than a few billion. Have the Hollywood scriptwriters made a mistake? No, the error lies in the translation: in English-speaking countries a billion is a thousand million, and for us a billion is a million million. Likewise, they interpret trillions differently.

Name	*Anglo-Saxon value*	*International value*
Billion	1 000 000 000	1 000 000 000 000
Trillion	1 000 000 000 000	1 000 000 000 000 000 000

Within the American space program, one of the projects that caused the most stir at the end of the 20th century was the investigation of Mars. The Pathfinder (1996) and Mars Global Surveyor (1996) missions were followed by Deep Space I (1998), Mars Climate Orbiter (1998) and finally Deep Space II (1999) and Mars Polar Lander (1998). Of these missions, which provided a great deal of information about the planet, two of them ended in a resounding failure due to technical failures: Mars Polar Lander crashed into the Martian surface when a problem with interference in the approach sensors turned off the braking engines during the descent phase; Mars Climate Orbiter (MCO) had a more curious ending.

This space probe was built by the American aerospace company Lockheed Martin. As is customary in that country, the company used English units of measurement, such as miles, in the programming of the probe, without anyone realizing that the company they worked for, NASA, uses the metric system for all its measurements and calculations. The space agency also did not notice the error until the craft was already in the vicinity of Mars. The result is well known to everyone: the probe did not enter Martian orbit, it simply ended up crashing into the planet.

But in the face of these problems, why doesn't everyone adopt the same units? To achieve this, the International Metric System was created in the 18th century. However, before explaining this system, let's go back a few centuries to see what the situation was with units of measurement.

5.2. The old measures

As an example, the following table is presented, which is a very summarized extract from a manual from 1874.

Table 2.4. Old measurements and their location

Location	*Unit*[†]	*Equivalency*
Barcelona	1 cane	1.55 metres
	1 metre	5.14 hand spans
	1 pound	0.40 kilograms
	1 medicinal pound	0.3 kilograms
	1 barrel	30.35 litres
	1 litre	1.05 mitadellas [half measures]
	1 quart of oil	4.15 litres
	1 litre of grain	0.17 quarts
	1 half-quarter for aggregates	34.75 litres
	1 land-area	41.22 square canes
Bilbao	1 media azumbre*	1.11 litres
	*half a ‘generous’ measure	
	1 media arroba de aceite*	6.74 litres
	*1 half of a ‘quarter’ [of a quintal = an average donkey load] of oil	
Burgos	1 pound	0.46 kilograms
Caceres	1 pound	0.45 kilograms
Lleida	1 [cántaro] jug of wine	11.38 litres
	1 litre	1.05 porrones [jugs]
	1 litre of grain	1.30 picolines [bushels]
	1 land-area	41.19 square canes
León	1 media cántara [1 half a pitcher]	7.09 litres
Madrid	1 half bushel for aggregates	27.67 litres
	1 litre of grain	0.86 quarts
Malaga	1 half bushel for aggregates	26.97 litres
Orense	1 cántara [a jug]	15.96 litres
Oviedo	1 cántara [a jug]	18.41 litres
	1 half bushel for aggregates	37.07 litres
	1 litre of grain	1.72 quarts
	1 ox day [ploughed]	12.57 land-areas (1,800 square rods)
Pamplona	1 robo para aridos [bushel]	28.13 litres
Pontavedra	1 ‘ferrado’ for measuring corn	0.57 ‘concas’ [sometimes ‘basin’]
Tarragona	1 half cane	0.78 metres
	1 metre	5.12 hand spans
	1 ‘ermine’ for liquids	34.66 litres
	1 ‘sinquena’ for oil	20.65 litres
	1 half-quarter for aggregates	35.40 litres
	1 litre of grain	0.16 quarts
	1 land-area	41.58 square canes
Teruel	1 pound	0.36 kilograms
	1 half [cántaro] jug of wine	10.96 litres

[†] the unit names have been assigned by translation to English and not by the defined measures in English

Although the ‘barrel’ and the ‘gravel’ measured different products, it can be seen that the amount represented is similar. These examples give an idea of how difficult it must have been to trade with a new city or with a new product. To this we must add that the same measures in nearby places had different values:

Barcelona	1 litre of grain	0.17 quarts
Tarragona	1 litre of grain	0.16 quarts

Orense	1 cántara [jug]	15.96 litres
Orviedo	1 cántara [jug]	18.41 litres

To give you a slight idea of how the same unit could vary depending on where it was used, we show the following table:

Table 2.5. Variation of old measurements in Spain.

	Unit	*Equivalence*	*Place of use*
Length	foot	0.259 m to 0.309 m	Catalonia, Balearics, Valencia, Castile
	rod [or cane]	0.772 m to 0.839 m	All of Spain
Mass	‘arroba’	11.5 kg to 12.5 kg	Castile, Galicia
	pound	372 g to 579 g	Almost all of Spain
	ounce	23 g to 37 g	the same areas as the pound
Capacity	‘azumbre’	0.66 l to 2.016 l	Castile, Galicia, Biscay
	‘cántara’ [jug]	approx. 16.13 l	All of Spain
	‘cañado’	32 l to 37 l	Galicia
	‘fanega’ [bushel]	10 l to 55.5 l	All of Spain except for Galicia, Leon and Navarre
	‘cargas’ [load]	121.6 l to 222 l (between 3 and 4 ‘fanega’)	The same areas as the ‘fanega’ [bushel]
	pint	0.735 l to 0.808 l	Aragon, Navarre

When it comes to currency, the situation becomes much more complicated. This is undoubtedly the unit of currency that has undergone the most changes over time.

To simplify exchanges, the monetary unit, which was the real, was defined with the following equivalences:

1 escudo = 10 reales
1 Isabel doubloon = 100 reales
1 real = 10 tenths
1 half reale = 5 tenths
1 double tenth = 2 tenths
1 tenth reale (the unit)
1 half tenth = 5 hundredths

We do not know the origin of many measures, but the best known have their immediate origin in the ancient Romans. They, like other ancient peoples, quantified their environment in relation to the dimensions of their body and the duration of natural cycles. Thus, to measure lengths they had the foot (pes) which is equivalent to 29.58 cm. Each foot was divided into 12 inches (unciae), which is the length of the distal phalanx of the thumb; 5 feet were 1 step (passus), 1.479 meters, which is what is covered with a double stride; 10 feet were equivalent to 1 perticae (perticae), 2.957 meters; 120 feet (12 perticae), 1 actus vorsus, which literally means the length of 1 ploughshare; and 1,000 steps, 1 Roman mile (i.e., 1,479 meters), equivalent to 8 Greek stadia. The foot could be divided into digiti, the width of a finger, and into four palmi, palms of 7.4 cm, which was the distance between the base of the index finger and the little finger, with the hand closed.

There were other traditional Egyptian measurements such as the palmipes, 36 cm; the cubitus or ulna, 44 cm, which was the length of the forearm from the end of the middle finger to the elbow, and the gradus, 73 cm.

Among all the other measures they used, we will highlight, for its durability until today, the Roman pound, of 327.45 grams, and the ounce (uncia), which was the twelfth part of a Roman pound.

5.3 The mile

As picturesque as its origin is its evolution. Several units have emerged from a single unit, all of them called "miles", with different values according to their use and the time. The current English mile was defined during the reign of Elizabeth I. It was established that it should consist of 8 furlongs, of 40 perches of 16.5 feet each, that is, 1,760 yards of 3 feet each.

In addition, we have the nautical mile, which is used to measure distances and speeds at sea. It is defined as the length of the minute of circumference of the great circle of the Earth (which, if we assume it is perfectly spherical, gives a great circle of 5,400 millias or 10,000 km), that is, 1,851.852 meters; or as the minute of meridian measured at longitude 45°, that is, 1,851.83 meters. For all these reasons, the value of the nautical mile is taken as:

1 nautical mile = 1,852 meters

The units related to the nautical mile are:

3 nautical miles = 1 nautical league
1 mile = 3 cables of 185.2 meters
1 cable = 111 fathoms

In the US, the standard geographic mile is the length of one minute of latitude of a sphere whose surface is assumed to be equal to that of the Earth. This gives a mile of 6,080.27 feet, which contrasts with the common mile, also in the US, which is 5,280 feet. Not to be outdone, they also have their own nautical mile, which is equal to 1,853.24 meters.

And to accentuate the Anglo-Saxon idiosyncrasy, its admiralty takes the mile of 6,080 feet, in order to arrive sooner to the battles!

We will show the equivalence between the different miles that have been in use:

Table 2.6. Types of miles

Type of mile	*English miles (legal)*
German short	3.897
German geographical	4.611
German long	5.753
Austrian	4.714
Danish	4.684
Old Scottish	1.127
English (geographical)	1.153
Irish	1.273
Portuguese	1.296
Prussian	4.680
Swedish	6.648
Swiss	5.201
Kilometre	0.621

Other units of the Anglo-Saxon system are:

Table 2.7. Anglo-Saxon measurements

Name	*Value*
1 inch	2.54 cm
1 foot	12 inches
1 yard = 3 feet	91.44 cm
1 fathom = 2 yards	1.829 m
1 mile (on land) = 1,760 yards	1.609 km
1 nautical mile = 6,080 feet	1.853 km
1 ounce	28.3 g
1 pound	454 g
1 pint	0.5683 l
1 gallon = 8 pints	4.546 l
1 acre	0.4047 ha

5.4. The Decimal Metric System

In 1790, shortly after the French Revolution (1789), the National Assembly agreed to create a new system of weights and measures. In March 1791, they chose the term "metre" as the name for the basic unit of measurement, a word that comes from the Greek *metron*, meaning "measure."

On 10 December 1799, the French Executive Council decreed new values for weights and measures based on the metre. These values were accepted by many countries throughout the 19th century, although for a long time most of them continued to use traditional measurement systems in everyday life. These new values were officially adopted in Germany in 1872 and in Spain in 1875.

At that time the metric system was defined with five simple units:

- Metre: unit of length equal to one ten-millionth of a quadrant of the meridian from the North Pole to the equator. It is equivalent to approximately 3 feet and 7 inches.

- Are: an area unit equal to 100 square meters. 100 ares are 1 hectare and 1 are is equal to 100 centiares.
- Cubic meter: unit of volume of solids.
- Litre: unit of capacity equal to 1 cubic decimeter.
- Kilogram: unit of weight equal to 1 cubic decimeter of distilled water at 4 °C in vacuum. It has multiples of the metric quintal (100 kg) and the ton (1,000 kg); lower units are the hectogram (0.1 kg), the decagram (0.01 kg) and the gram (0.001 kg).

To name other multiples and submultiples of the basic units of the metric system, one of the following prefixes must be added before the name of the unit:

Table 2.8. Metric Decimal System Prefixes

	Prefix names	*Prefix characters*	*Values*
multiples	myria-	my	10,000
	kilo-	k	1,000
	hecto-	h	100
	deca-	da	10
submultiples	deci-	d	0.1
	centi-	c	0.01
	milli-	m	0.001

5.4.1. The metre

As we have already pointed out, the meüo was originally defined as one ten-millionth of the quadrant of the Earth's meridian. This definition was chosen because at that time topography was at its peak and extensive knowledge had already been acquired about how to measure distances on the Earth's surface.

However, the difficulty of measuring the entire quadrant was very great, so it was decided to measure only a fairly large arc, one end of which was to the north and the other to the south of the 45th parallel north. The part of the meridian between Dunkirk and Barcelona was chosen, the ends of which are at sea level and the arc passes through Paris (let us not forget that these measurements were promoted by the French).

Over the years, the definitions of the five basic units would change, becoming more precise and at the same time adapting to the new times. As can be seen, although the attempt to define the units was good, it suffered from certain problems. For example, the metre was defined in relation to something (in relation to a quadrant of the meridian), the determination of which was inexact, despite the fact that the topography was very advanced; also the kilogram, initially defined as a unit of weight, was later redefined as a unit of mass (the unit of weight is the *kilopond* and it is the force with which the standard gravity of the Earth attracts a mass of one kilogram). There were other differences with respect to the current system: the prefixes for multiples and submultiples were not used for all units (for example, the multiples of the kilogram were the quintal and the ton); The unit of surface was the ‘are’ rather than the square meter, and the cubic meter was defined as a unit of volume or solidity (whatever they intended to define by solidity).

Despite its problems, this new system of measurements represented a great advance because, at that time, each human group had its own system of units, that is, each state, kingdom, county, city and town measured with its own units, which is why disputes regarding the measurement of magnitudes (quantities, payments, etc.) were frequent.

The meter has had the following definitions:

1. The 1/10,000,000th part of the quadrant of the terrestrial meridian that passes through Paris.
2. The length between the notches of the iridium platinum bar stored in Sévres, measured at 0 °C.
3. The length of 165,076,373 wavelengths in vacuum of the radiation corresponding to the transition between the “$2p_{10}$ and $5d_5$ orbitals of the krypton-86 atom.”
4. The length of the path of a ray of light in vacuum in a time interval of 1/299,792,458 seconds.

5.5. The International System of Units

In 1960, the so-called International System of Units was adopted, which aimed to update and universalize the measurement system. To this end, the basic units of fundamental magnitudes were defined, which were established in these seven:

Table 2.9. International System of Units

Quantity	*Name*	*Symbol*	*Definition*[†]
Length	metre	m	Distance travelled by light in a vacuum in 1/299,792,458 seconds.
Mass	kilogram	kg	Mass of the platinum-iridium standard cylinder kept in Sèvres (France). It is the only one defined with respect to an object and with a prefix (kilo).
Time	second	s	Duration of 9,192,631,770 periods of the radiation corresponding to the transition between two hyperfine levels of the ground state of the caesium 133 atom.
Electric Current	ampere	A	Intensity of a constant current which, maintained in two parallel, straight conductors, of infinite length, of negligible circular section and located at a distance of one meter from each other in a vacuum, produces a force equal to 2×10^{-7} newtons per meter of length.
Temperature	kelvin	K	1/273.16 parts of the thermodynamic temperature of the triple point of water.[1]
Amount of Substance	mole	mol	Quantity of substance in a system that contains as many elementary entities as there are atoms in 0.012 kilograms of carbon 12.
Luminous Intensity	candela	cd	Intensity, in a given direction, of a source that emits monochromatic radiation of frequency 540×10^{12} hertz and whose energy intensity in said direction is 1/683 watts per steradian.

[1] The triple point of water represents the conditions of pressure, volume and temperature that make it possible for the three states (solid, liquid and gas) to coexist simultaneously.

[†] [The text is translated from the author’s Spanish rather than any official definition.]

The International System of Units also defines:

- Supplementary units: radian, steradian.
- Derived units: farad, hertz, joule, newton, ohm, pascal, volt, watt, coulomb, weber, tesla, henry.

The use of multiples and submultiples can be represented by the values shown in the following table:

Table 2.10. Prefixes of the International System.

Prefix	*Symbol*	*Value*	*Powers of Ten Equivalent*
yotta	Y	1,000,000,000,000,000,000,000,000	10^{24}
zetta	Z	1,000,000,000,000,000,000,000	10^{21}
exa	E	1,000,000,000,000,000,000	10^{18}
peta	P	1,000,000,000,000,000	10^{15}
tera	T	1,000,000,000,000	10^{12}
giga	G	1,000,000,000	10^{9}
mega	M	1,000,000	10^{6}
kilo	k	1,000	10^{3}
hecto	h	100	10^{2}
deca	da	10	10^{1}
-	-	1	10^{0}
deci	d	0.1	10^{-1}
centi	c	0.01	10^{-2}
milli	m	0.001	10^{-3}
micro	µ	0.000 001	10^{-6}
nano	n	0.000 000 001	10^{-9}
pico	p	0.000 000 000 001	10^{-12}
femto	f	0.000 000 000 000 001	10^{-15}
atto	a	0.000 000 000 000 000 001	10^{-18}
zepto	z	0.000 000 000 000 000 000 001	10^{-21}
yocto	y	0.000 000 000 000 000 000 000 001	10^{-24}

5.6. New units

With the arrival of new technologies and new areas of knowledge—such as computing—it was necessary to introduce new units or adapt old ones to the circumstances existing at the beginning of the 21st century.

In the case of computing, it was necessary to adapt the entire algebraic system (numbers, mathematical operations and algebraic relationships), which generally use the decimal system, to the binary system, which consists of only two symbols, 0 and 1.

Table 2.11. Hexadecimal binary system.

Decimal	*Binary*	*Hexadecimal*	*Decimal*	*Binary*	*Hexadecimal*
0	0	0	8	1000	8
1	1	1	9	1001	9
2	10	2	10	1010	A
3	11	3	11	1011	B
4	100	4	12	1100	C
5	101	5	13	1101	D
6	110	6	14	1110	E
7	111	7	15	1111	F

The table above shows the relationship between the decimal and binary systems. The hexadecimal system is also shown, with its digits from 0 to 9 and from A to F, since this is also often used in computing to give a binary value in a compacted way, since every four binary digits forms a hexadecimal digit.

So, for example:

1b + 1b = 10b
101b + 11b = 1000b
1010b = 10d = Ah

where ‘b’ is binary, ‘d’ decimal and ‘h’ hexadecimal.

In computing, each of the two binary values, 0 and 1, is called a "bit". A bit is the fundamental unit of information. It has no submultiples, and its multiples are as shown in the following table: [Because the SI unit prefixes are referenced to the powers of ten for all other uses it has become necessary to reconsider the application to description of powers of two. The reconsideration is included in the table.]

Table 2.12. Bit multiples

Informal names	*Standard names*	*Symbols*	*Values*	*Powers of 2*
bit	bit	b	1	0
kilobit	kibibit	kib	1024	10
megabit	mebibit	Mib	1024 kib = 1,048,576	20
gigabit	gibibit	Gib	1024 Mib	30
terabit	tebibit	Tib	1024 Gib	40

As can be seen, the multiples are powers of 2 and not 10, as is the case with the units of the International System [see the note to the table reference above]:

1 kib = 2^{10} bits; 1 Mib = 2^{20} bits = 2^{10} kib

There are other multiples that are given not by the quantity they represent, but by the number of bits they use:

Table 2.13. Other bit multiples.

Name	*Number of Bits*
Nibble	4 bits e.g. 1010
Byte	8 bits e.g. 10101010
Word	Variable: 4, 8, 16, 32, 64, 128

Of these multiples, the most commonly used is the byte, which can use the same prefixes as the bit to form its multiples (informally kilobyte, gigabyte, megabyte, terabyte).

For example: 1 kilobyte = = 1 kibibyte = 1,024 bytes.

The term "word" is used to indicate the number of bits that can be processed simultaneously, which is why its meaning will depend on the way in which the information is processed or the processing capacity of the microprocessor we use.

6. Is Hollywood deceiving us? Distortions between fiction and reality

You may have heard someone commenting on the way out of the cinema like: “This only happens in the movies.” And it’s true, because many of the situations shown on the screen do not correspond to reality. Mathematics and physics can explain these wonders of the seventh art.
In the film *Star Wars* we hear the phrase: "Open fire when you see the target," and then we see the launch of a destroyer missile. Today's technology allows us to destroy targets without having to have them in sight, so the previous order is totally out of place in such a technologically advanced ship. But that's not all: physics teaches us that explosions in space do not make noise because there is only vacuum and so no medium to carry sound, contrary to what the film shows us.

Another common scene in Hollywood movies is the explosion of a car caused by a bullet hitting its gas tank or by a collision with another vehicle. But such an explosion is only possible when the tank is a quarter full and the gasoline vapours come into contact with a flame; but even under these conditions it would be unlikely that the car would explode.

Figure 2.9. Image from the film The Peacemaker *(1997)*

Speaking of impossible situations, what about the classic killer who hides between the front and back seats of a vehicle, and who curiously always goes unnoticed by the victim? Well, more than a killer, he is a contortionist, since the dimensions of this space are usually 100×30×45 cm. The viewer can be fooled by the angle offered by the camera, which does not allow him to appreciate the lack of space.

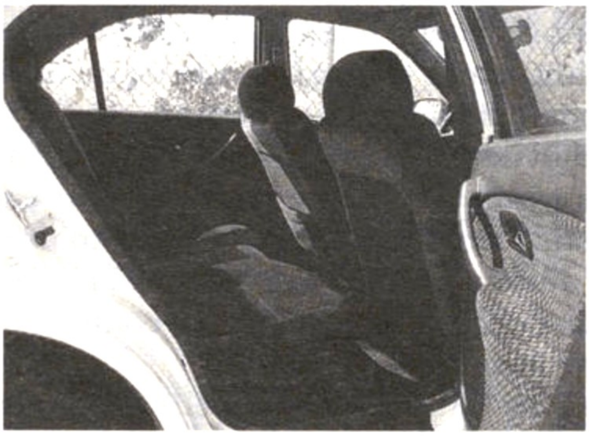


Figure 2.10. Interior space of a vehicle

Gunfights are also very common on screen. They usually occur at speeds of over 100 km/h and with constant lane changes. Mathematics shows that the chances of success are one in a hundred thousand.

Speaking of gunfire, let us recall Rambo's famous machine gun, which fires at 800 rounds per minute. Its magazines hold 100 rounds and the barrel needs to be changed every 10,000 rounds. Considering that in the film the magazine is changed in less than 3 minutes (which is contradicted by reality), Rambo would have died very soon.

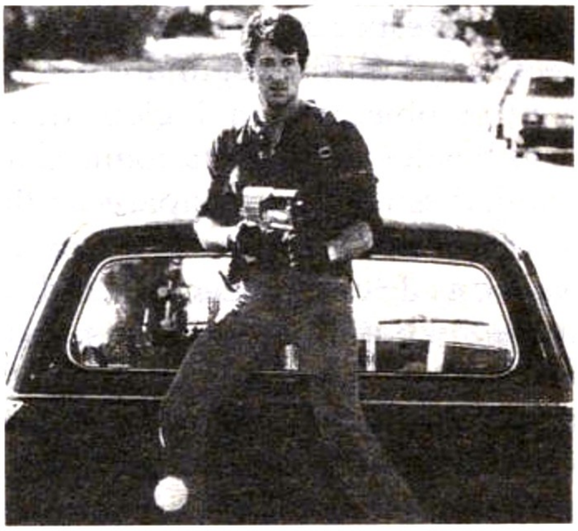


Figure 2.11. Image from the movie Rambo

Let's continue with the subject of shooting, but let's move on to another category. In fiction we can see how missiles are fired from an airplane at rest. But in reality these weapons cannot be activated until the plane has taken off and its landing gear has left the ground.

Now let's go out into space. Mathematics and physics show that bodies exposed to the vacuum of space do not deform or explode, contrary to what the film **“Outland”** (1981), starring Sean Connery, presents us. A human organism exposed to the lack of atmosphere would die by drowning or freezing, but would never explode.

Another aberration of cinematic physics comes from *Independence Day*, where, in the middle of a wave of fire launched by the enemies and which runs through the 2nd Street Tunnel in Los Angeles, the protagonist takes refuge in a toilet and emerges unharmed. But in the sad reality, the wave of fire would have consumed all the oxygen and our protagonist would have suffocated.

Time bombs always have a clock attached to them that tells the time left until the explosion. In real life, only the person who made the device knows when it will explode. Real bombs lack this warning device, which is only used in movies to make the viewer suffer until the protagonist disconnects it at the last second.

But there are other technological objects that are used differently on the big screen than in reality. Sometimes two people are shown talking simultaneously on a walkie-talkie. But this device is a one-way communication device; you have to keep the button pressed while you are talking and release it to hear the other person.

In many films we can also find the following comical situation: a police station receives a call from a criminal, usually a psychopath or a kidnapper. The police officers try to have a conversation with the alleged "bad guy" so that their colleagues have time to trace the call, but suddenly the call is cut off, the location is unsuccessful and the suspense manages to extend until the end of the film. It should be said that for years there has been a device called "Caller ID" that allows, in real time, to know the place from which the call is being made. This device works on mobile phones and landlines that show the interlocutor's number on the screen.
Another manipulation of reality can be found in the film *Mission: Impossible* at the end we witness a spectacular chase with a fight between the protagonists on top of the high-speed train that links France with England through the Channel Tunnel. A helicopter is attached to the train, which is forced to manoeuvre to avoid another train coming towards them. The scene is a series of absurdities: it is impossible for two people to fight on top of a train travelling at more than 300 km/h; this train runs on electricity through a line whose overhead lines would not allow the helicopter to pass; the Eurotunnel is 7.6 m high, a space too small for a helicopter to manoeuvre inside; in addition, the crossing of the two trains inside the Channel Tunnel is impossible because there is a different tunnel for each of the two directions of traffic.

We can also talk about King Kong, the 2,900-kilogram, 14.5-meter-tall giant ape, based on a real gorilla weighing 230 kilos and 1.80 meters. Mathematics allows us to see, thanks to Euclidean geometry, that the change of scale is not correct. According to Euclid, if two figures are proportional and to scale, it is verified that if the quotient of their lengths is a value a, then the quotient of their areas takes the value a^2 and the quotient of their volumes is a^3; the factor a is called the "proportionality factor." If we apply this rule to our King Kong, we obtain the following result:

Height of the movie monster / height of the chosen gorilla = 14.5 / 1.8,
which is approximately 8.

According to the result of Euclid's geometry, the quotient of the volumes would have to be:

$8^3 = 512$

We see that:

Mass of King Kong / mass of the original gorilla = 2,900/230 = 12.6

This result shows that our King Kong is not made to scale. If we wanted to respect the height and volume data of the chosen gorilla, the gorilla in the film would have to weigh (512×230) approximately 120 tons.

A monster of this size would hardly be agile enough to climb the Empire State Building; it couldn't even stand upright, proving that the movie *King Kong* is pure science fiction. For reference, *Tyrannosaurus rex*, the one of the largest known bipedal animal to have ever lived on Earth, weighed only up to 9 tonnes.

Figure 2.12. Image from the film King Kong (1933)

It is not possible to end the review of cinematography without mentioning the legendary film Superman, whose hero protects humanity by stopping bullets with his hand, flying and running at high speeds. The film attributes his powers to the difference in environmental conditions between the planet Krypton (where Superman comes from) and Earth. Mathematics and physics can show us that Superman, even with magical powers, is incapable of stopping a 50,000 kg truck with his hand without travelling a distance backwards of at least 25,000 metres. To obtain this distance we have applied Newton's second law:

Force = Superman's mass × acceleration = 90 kg x 9.81 m/s^2 = 883 newtons

Acceleration of braking on the truck
= force / mass of the truck = 883 newtons / 50,000 kg = 0.018 m/s^2

According to physics:

Distance = speed^2 / (2 × acceleration) = (30 x 30) / (2 x 0.081) = 25,000 meters

In the film, Superman only travels 1 meter.

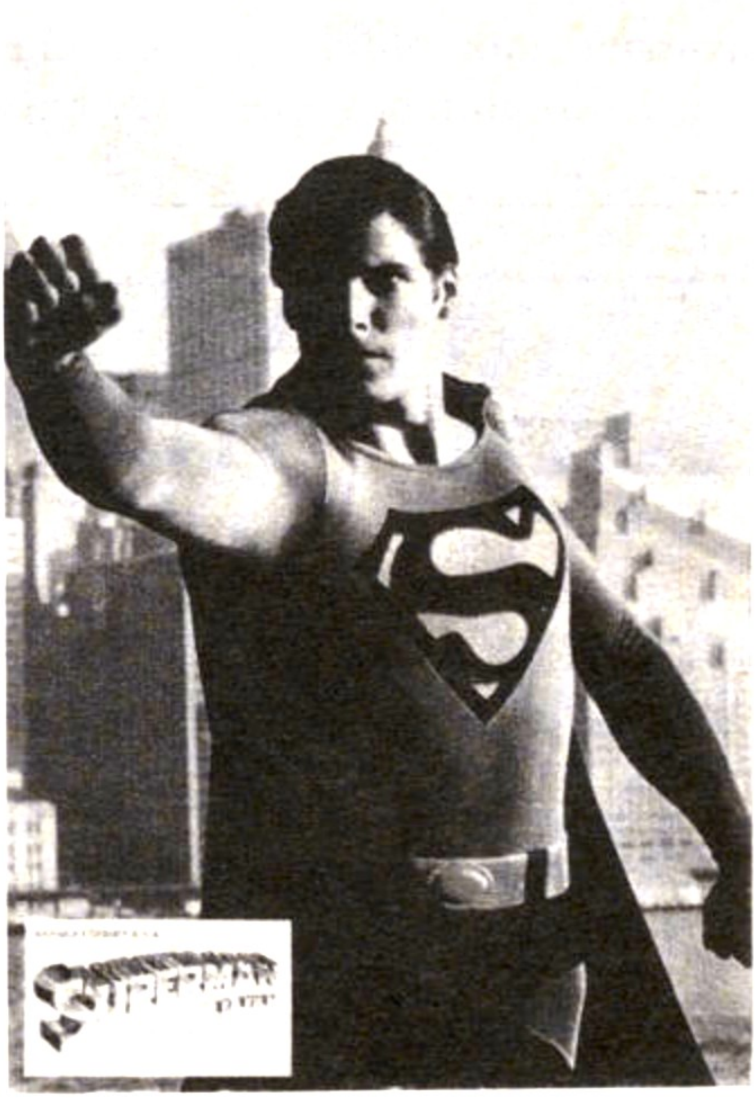


Figure 2.13. Image from the movie Superman, the film (1978)

From all this we can conclude that mathematics can help us to distinguish manipulation from reality. In the cinema, the laws of nature are not treated with sufficient scientific rigor, which often confuses the viewer. But mathematics, physics and technology offer us the tools and knowledge necessary to distinguish what can and cannot happen off-screen.

Mathematics plays an important role in current technical development and can offer specific and interesting content to students; it is enough to show them this “other side”, coordinating the formative and informative character. Mathematics can contribute to enhancing imagination, creativity, critical faculties and intelligent dialogue, and to developing an education in line with the new millennium.

III. THE GOLDEN NUMBER: ART AND NATURE

Geometry has two great treasures: one is the theorem of Pythagoras; the other, the division of a line into extreme and mean ratio. The first we may compare to a measure of gold, the second we may name a precious jewel.

Kepler (1571-1630)

1. The golden rectangle

1.1 Introduction

I once showed a group of students a collection of four rectangles so they could choose the one they thought was most pleasing, most beautiful. These rectangles can be seen in the illustration.

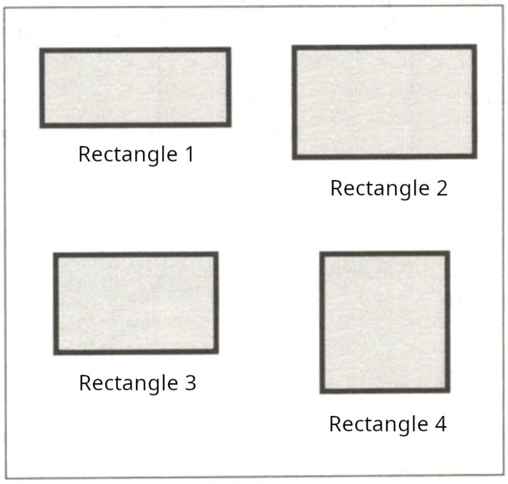


Figure 3.1. Choosing rectangles

The result was that a large proportion of respondents opted for rectangle number 2. But what is so special about this rectangle?

Let's look at the length of its sides: 2.5 cm × 1.545 cm

Its ratio is: $\frac{2.5}{1.545} = 1.618 = \Phi$

Rectangles whose ratio between sides is 1.618 (the golden ratio) are called "golden rectangles."

In various stages of nature and art, mathematics plays a prominent role, especially in regard to the golden ratio. We will focus on this ratio and its influence on everyday life.

This attractive property of the golden rectangle has been widely used throughout human history, from Antiquity to Romanesque, Renaissance and even the present day, as we will see later. First, we will show how the golden rectangle is geometrically constructed.

1.2. Graphic construction of a golden rectangle

We invite the reader to do the following exercise:

We draw a square and mark the midpoint of one of its sides. We join it with one of the vertices of the opposite side and transfer this distance to the initial side (we obtain AC); in this way we obtain the longest side of the rectangle.

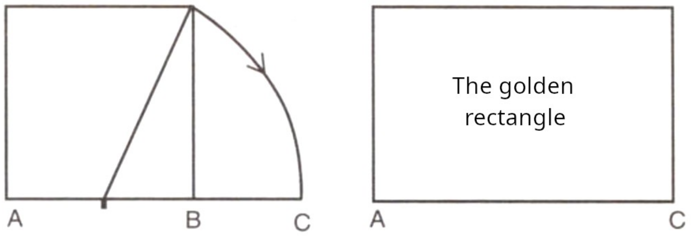


Figure 3.2. Graphic construction of a golden rectangle.

If the side of the square is equal to 2 units, the larger side of the rectangle is equal to: $1+\sqrt{5}$,
so the proportion between the two sides is: $(1+\sqrt{5})/2$

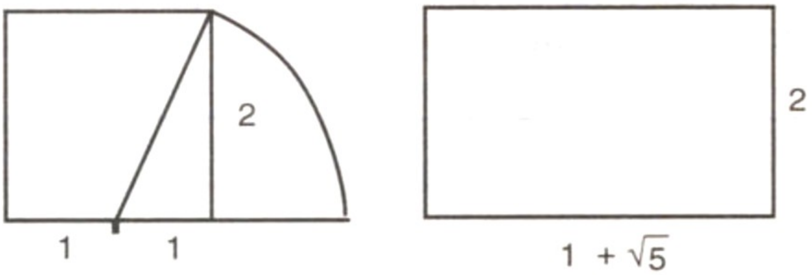
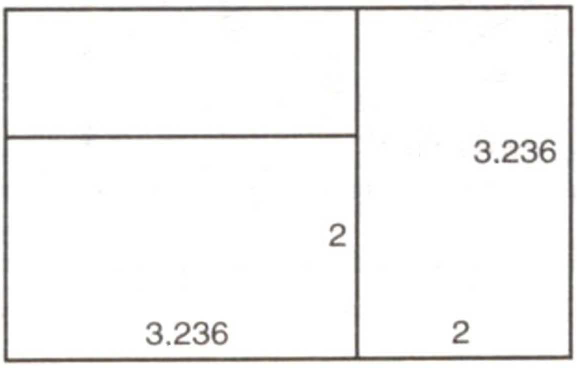


Figure 3.3. Proportion between the sides of a golden rectangle.

Another property of golden rectangles is that if we take two equal golden rectangles and place them as in the following figure, another larger golden rectangle is formed.



With two rectangles of 3.236×2 we form another of 5.236×3.236.

Figure 3.4. Composition of golden rectangles.

$$\frac{3.236}{2}=\frac{5.236}{3.236}=\Phi$$

1.3. The golden rectangle in art

Possibly due to the intrinsic attraction we feel for the proportions of the golden rectangle, it has been widely used in art. It is present in all periods and in all disciplines: painting, sculpture, architecture, and in both unknown and famous works. We find it in the pyramids, in the *Mona Lisa* (Leonardo da Vinci), *The Birth of Venus* (Sandro Botticelli), the *Venus de Milo* (Alexandros of Antioch) and the *Notre Dame de Paris Cathedral*. Let us look at some examples:

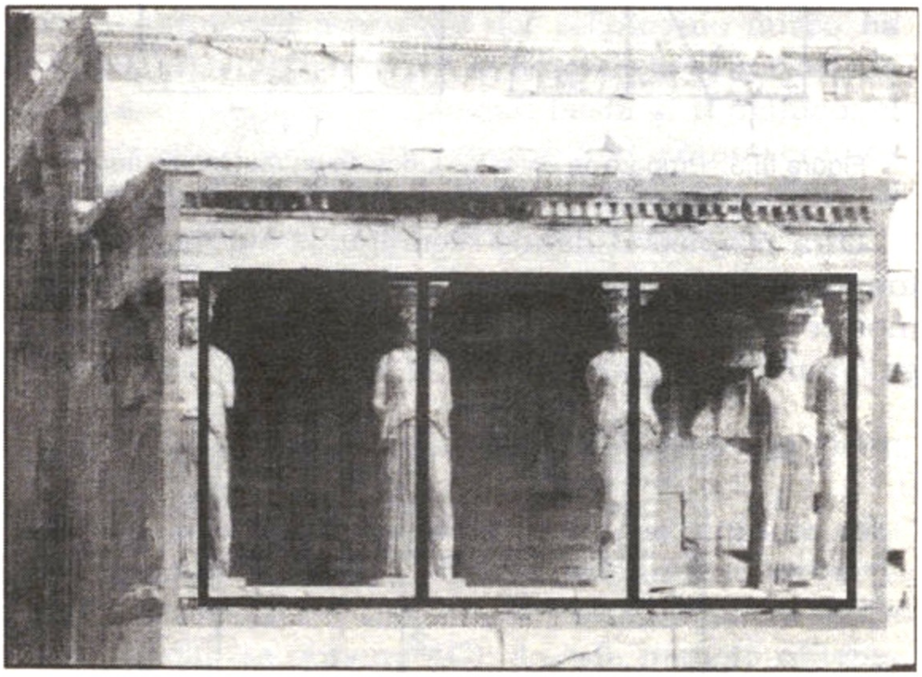


Figure 3.5. Porch of the Caryatids, on the south side of the Erechtheion.

In the Porch of the Caryatids, on the south side of the Erechtheion, we see how the columns form sides of golden rectangles. In turn, the façade is another golden rectangle.

Like the Caryatids, the Parthenon, on the Acropolis of Athens (447-438 BC), presents golden rectangles in various architectural elements.

The Vitruvian Man (Accademia Gallery, Venice), drawn in 1490 by Leonardo da Vinci, shows the proportions of the human figure. It is based on a passage by the famous Roman architect Vitruvius, in which he describes how the human form, seen from the front with hands and feet extended, can be inscribed within a circle with the navel as its centre and, in turn, enclosed within a square. In these circumstances the head is exactly one-tenth of the total body, and the ratio of the side of the square to the radius of the circle is the golden ratio.

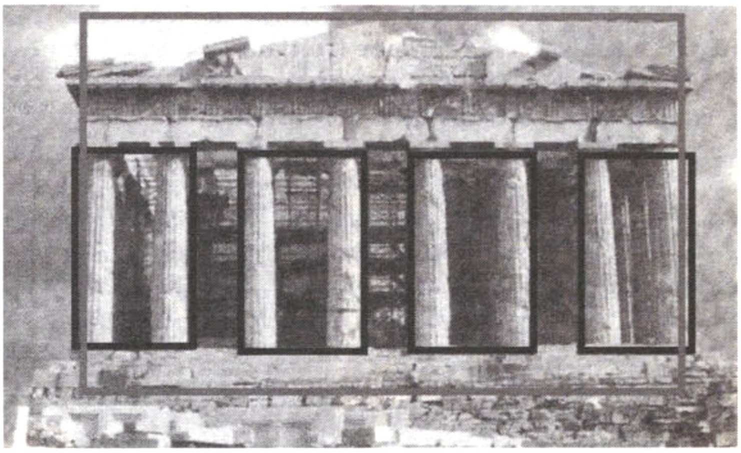


Figure 3.6. Parthenon in Athens

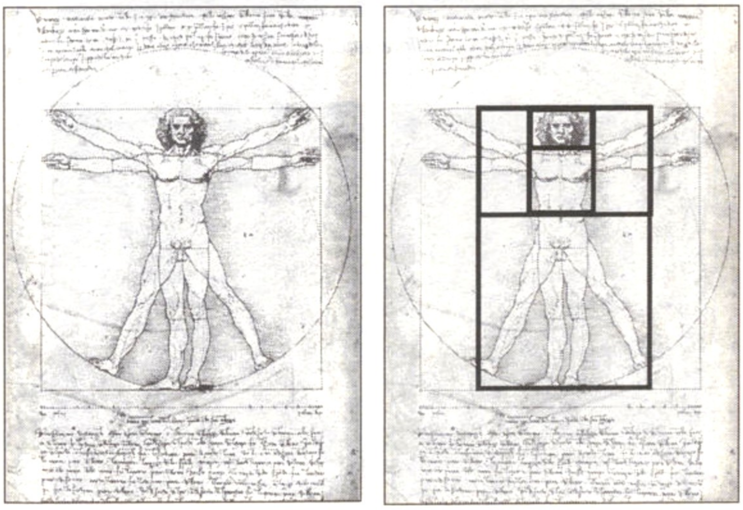


Figure 3.7. The Vitruvian Man, by Leonardo da Vinci

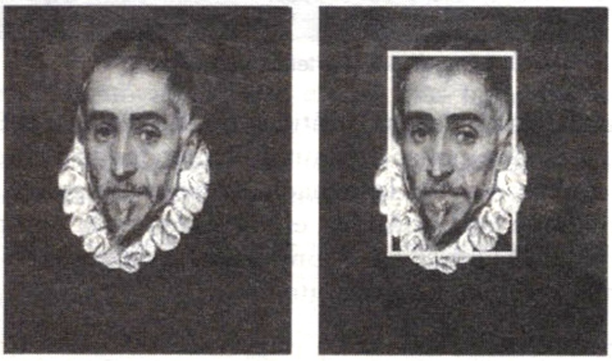


Figure 3.8. Elderly Gentleman, by El Greco.

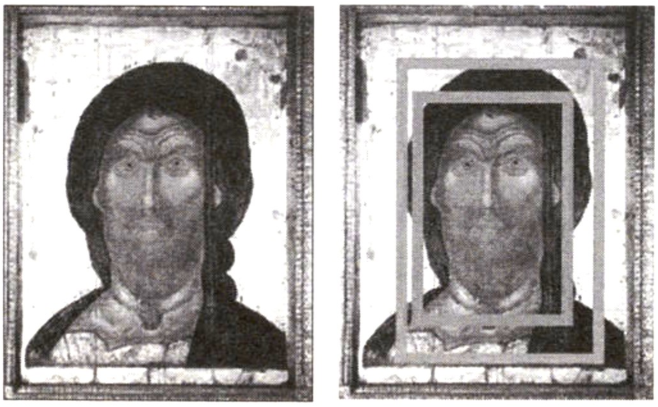


Figure 3.9. Christ in a 14th century Russian icon

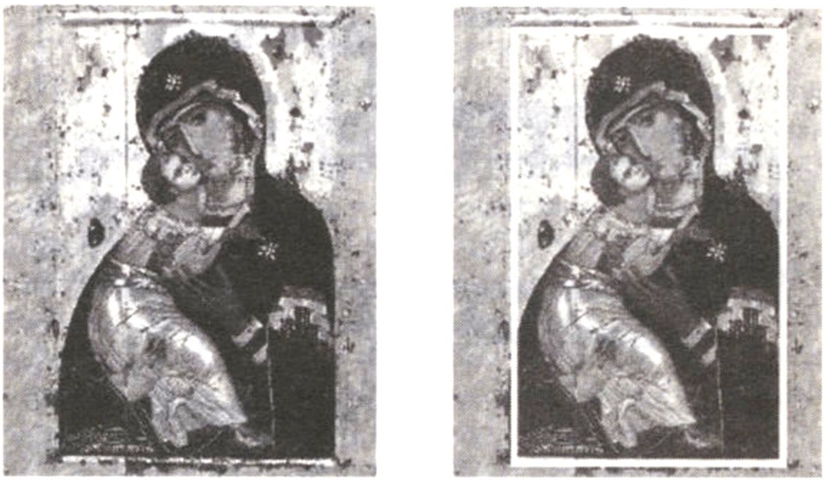


Figure 3.10. Russian icon: Our Lady of Vladimir, *11th century*

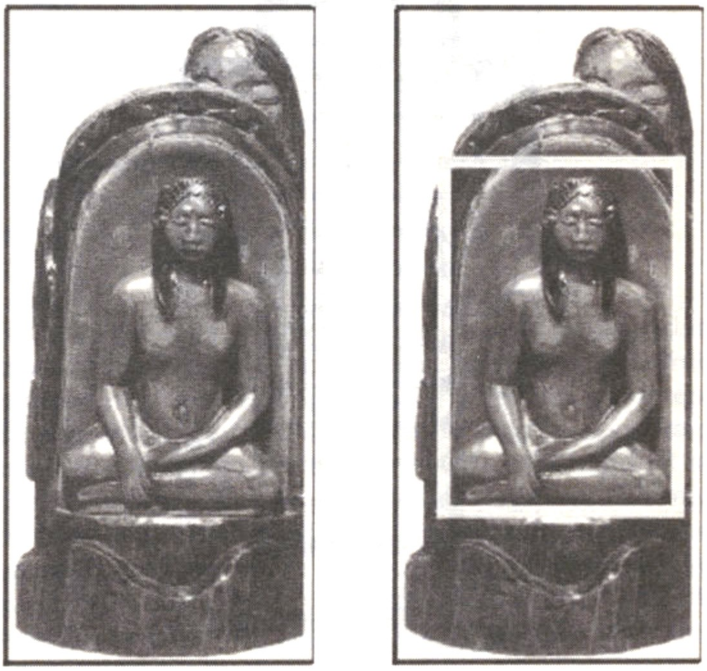


Figure 3.11. Idol of the Pearl, *by Gauguin*

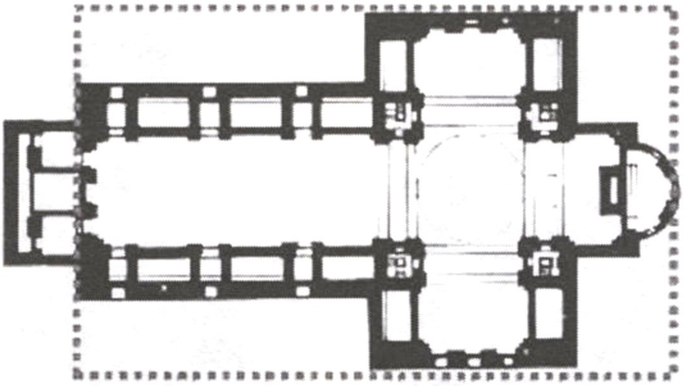


Figure 3.12. The plan of St. Andrew of Mantua

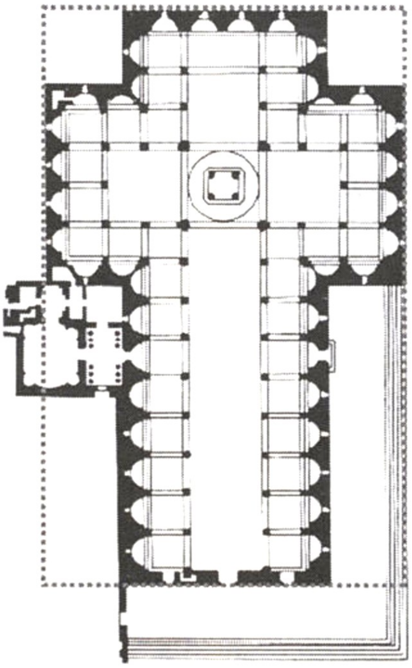


Figure 3.13. Plan of the Church of Santo Spirito (Florence)

1.4. The golden rectangle today

In a society like today, focused on materialism and productivity, it seems absurd to think that a figure like the golden rectangle could exist beyond museums and exhibitions. However, we are surrounded by golden rectangles. Why are many sports played on rectangular fields? They could be circular and then there would be no uncomfortable corner seats and the public could see clearly from any point. But no, they are rectangular. Of course, the measurements vary greatly from one sport to another, as they also depend on the number of players and their position on the field. For example, FIFA regulations (the international governing body for football) establish the following minimum and maximum measurements for a rectangular football field:

Minimum: 90 m × 45 m
Maximum: 120 m × 90 m

Its proportions would be in these cases:
Minimum: 2
Maximum: $1.\dot{3}$

The keen-minded will have noticed that these proportions only vaguely resemble a golden rectangle. But it should be noted that these are extreme cases, the minimum and maximum fields allowed; it is not very common to see fields of 90 × 45 or 120 × 90. If we choose a field of intermediate measurements (which are the most common), for example if we take a field of 115 × 68, we see that its proportion is 1.69. That's something else! In short: most football fields follow the golden ratio.

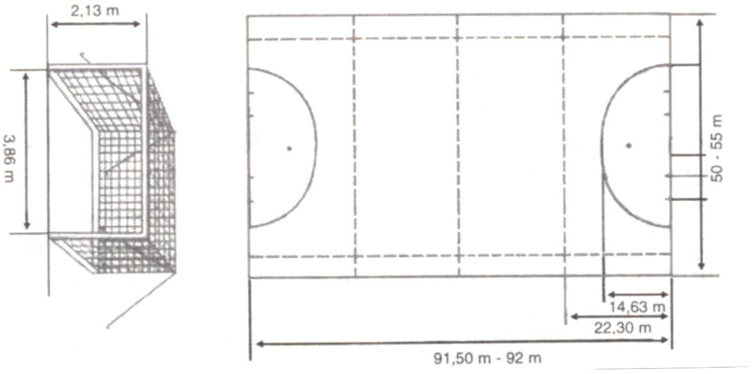


Figure 3.14. Dimensions of a hockey field

Rugby, basketball, hockey, etc. fields are in the same situation as football fields. All of them, apart from the charm of the sport, have an attractive setting: they are all golden rectangles. Citizens feel comfortable with these dimensions.

Let's look at another example. If you are a smoker, you can safely say that you are possessed by the "golden vice." Measure your pack of cigarettes... You have indeed found perfect golden proportions. In fact, a pack of cigarettes is a perfect design to induce tobacco consumption. Its shape, as we have said, is attractive to the eye, but also to the touch, since it fits the size of the hand and fits perfectly in a normal-sized pocket. And there is no need to talk about its contents, with substances that create addiction. A pack of cigarettes is the result of the most sophisticated sales engineering technique. This is how knowledge and beauty can also be used for diabolical purposes.

Fortuna
Las Autoridades Sanitarias advierten.
Fumar durante el embarazo daña al futuro hijo
ESPAÑA
DUCADOS
100% Tabaco
Natural
Las Autoridades Sanitarias advierten que
el tabaco perjudica seriamente la salud

Figure 3.15. Cigarette packs

However, you may not be a fan of

superficial and frivolous spectacles like football matches, or you may belong to the anti-smoking league. Even so, you will not escape the spell; you also carry a golden rectangle in your pocket, and if you don't believe it, look at your credit cards, perfect examples of golden rectangles.
This is when someone comes along and, with a smirk, says: “I don’t like sports, I don’t smoke, and I don’t have credit cards.” Bad luck again! Look at your wallet again and take out your ID card. Measure it and you’ll be surprised: the golden ratio appears!

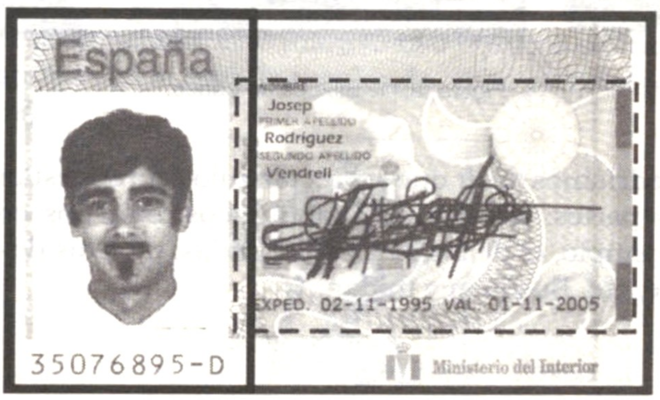


Figure 3.16. Identity document used in Spain

And, of course, in something as common as banknotes, the golden ratio could not be missing. Great care is taken to make them attractive and thus they become true works of art. Let us look at the already old banknotes used in the Spanish State, predecessors of the euro:

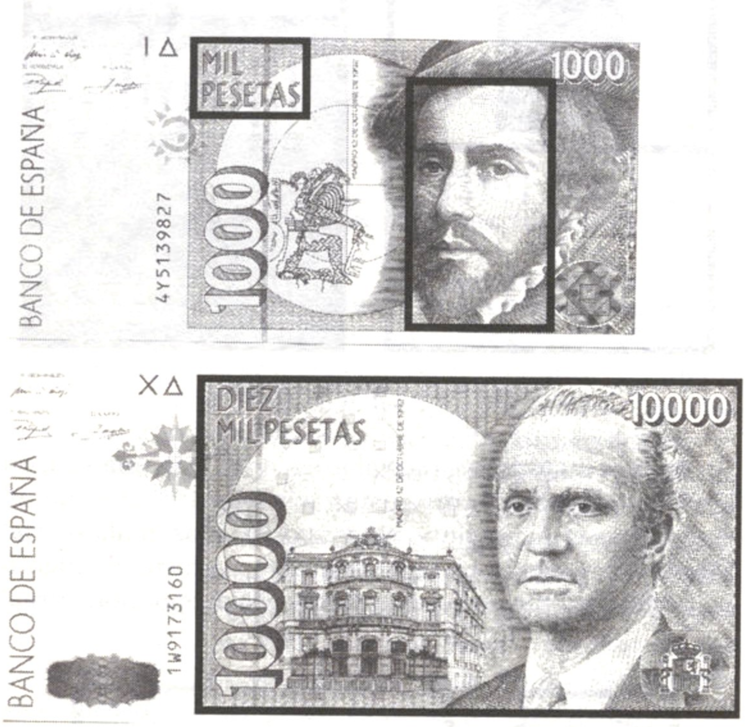


Figure 3.17. Spanish banknotes.

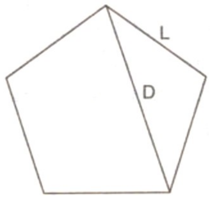
We could say that golden rectangles are everywhere: widescreen televisions, buildings such as the United Nations, chocolate bars, etc. Find and measure them.

1.5. Derivations and characterizations

This section is intended as an apologia for lovers of mathematics; the uninitiated reader may skip this section.

In this section we will show you the golden ratio from a mathematical point of view. You can do the following intellectual exercise:

Given a regular pentagon, we divide the length of any of its diagonals (D) by the length of one of its sides (L). Whatever the size of the pentagon, the value of the quotient is always 1.61803 (Φ phi, the golden number).



$$\boxed{\frac{L}{D} = 1.61803 = \Phi}$$

Figure 3.18. Φ*, Phi in the pentagon*

The presence of this wonderful number is also found in the regular five-pointed star (pentagram).

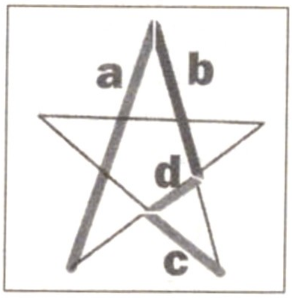


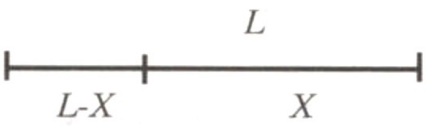
Figure 3.19. Φ*, Phi on the pentagram*

First of all, we will highlight the beauty of its mathematical properties. The following equalities are fulfilled in the pentagram:

$$c+d=b$$
$$c+b=a$$
$$\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=\Phi$$

Now suppose we have a segment of length L. We divide it into two parts and call *X* the larger portion and *L-X* the smaller. If we divide the total length of the segment by the larger portion and then divide the larger segment, *X*, by the smaller, *L-X*, we will see that for any value of *L* and *X* the two quotients are different except in one case: when *X*= 1.618 (Φ) and *L* = Φ + 1.

$$\frac{L}{X}\neq\frac{X}{L-X}$$



If $X = \Phi$ and $L\text{-}X = 1$, so $L = 1 + \Phi$, then:

$$\frac{L}{\Phi}=\frac{1+\Phi}{\Phi}=\frac{1}{\Phi}+1=\Phi$$

Expressed another way:

$$\frac{X+1}{X}=\frac{X}{1}$$

$$X^2=X+1$$

$$X^2-X-1=0$$

If we calculate the solutions of this equation, we have:

$$X_1=\frac{1-\sqrt{5}}{2}$$

$$\boxed{X_2=\frac{1+\sqrt{5}}{2}}$$

Definition of Φ, phi.

Of these two solutions, we will pay attention to the second one, that is, the golden number, since it has an extraordinary influence on everyday life.

Leaving aside the flat figures, we will highlight the formula to find the volume of the dodecahedron:

$$\boxed{volume=\frac{1}{2}\sqrt{5}\left(\frac{1+\sqrt{5}}{2}\right)^4\cdot a^3}$$

Volume of the dodecahedron with edge *a*

We see that the expression for the golden number appears clearly.

Studies on the golden number have given it numerous characterizations, among which we will highlight two, in which only the number 1 appears:

$$\boxed{\Phi=\lim\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}}}$$

Characterization of Φ as a limit

$$\Phi = 1 + \lim \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}}}$$

Defintion of Φ by the limit of a continued fraction

Phi, Φ, has also been characterized as a trigonometric function:

$$\Phi = 2 \cdot \cos\left(\frac{\pi}{5}\right)$$

Trigonometric definition of Φ

It can also be expressed as a limit of a recursive sequence:

$$X_N = \frac{X_{N-1}^2 + 1}{2 \cdot X_{N-1} - 1}$$

Iterative definition of Φ where Φ is its limit

W. A. Price, in a letter published in the English magazine *The Field*, asserts that, just as there can be only one triangle whose sides form an arithmetic progression 3, 4 and 5, and their similar ones, there exists only one triangle whose sides form a geometric progression: the right triangle with sides 1, √Φ and Φ. This triangle is called the "Price triangle". [There is no specific reference to this letter in the electronic databases of *The Field*. The right-angled triangle exists and is named after Johannes Kepler, the mathematician who worked on Tycho's data for Mars and deduced the rules of planetary motion called the Kepler Laws. Wikipedia cites a letter in 1575 written by Kepler as cited in a different author's work. It is called the Kepler triangle.]

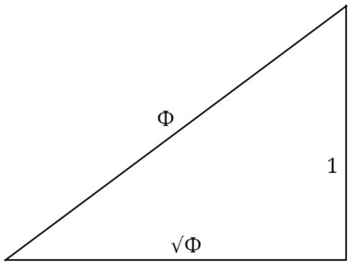


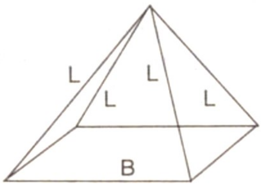
Figure 3.20 The Kepler Triangle

1.6. The golden ratio in history

In ancient civilizations, we often find the golden ratio. The base of the golden chamber that contained the tomb of Ramses IV is square and its height is in golden proportion to the side of the base. The plan of the temple of Osiris is a golden rectangle.

The pyramids, whose geometry contains knowledge of mathematics, physics, astronomy and other fields, have multiple references to the golden ratio. For example, in the pyramid of Cheops, the total area of the Great Pyramid is divided according to the golden section, such that the ratio between the total area and the lateral area is equal to the ratio between the latter and the area of the base.

Let's see it graphically:



$$\boxed{\frac{B+4\cdot L}{4\cdot L}=\frac{4\cdot L}{B}=\Phi}$$

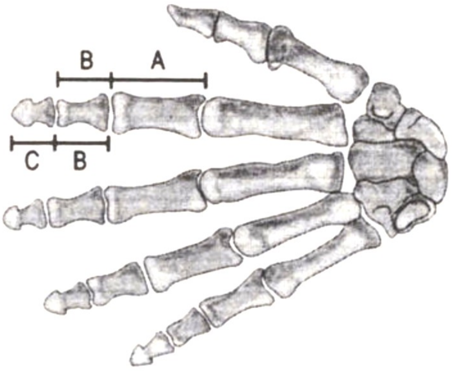
2. The golden ratio

2.1. The golden ratio and human physiology: The golden proportions in the human body

So far we have seen how the golden number is present in human society and culture. However, this presence is artificial; it has been imposed by individuals. However, our relationship with the golden number goes even further: we are closely linked to it naturally; from the moment we are born, our life runs parallel to this magical number.

Adolf Zeysing (24 September 1810 – 27 April 1876), a famous researcher of the golden ratio, made measurements on numerous human bodies and came to interesting conclusions. He found that the proportions of the male body are close to 13/8 and those of the female body to 8/5. We can observe, on the one hand, that both proportions are close to the golden ratio and, on the other, that their terms (13, 8 and 5) are numbers of the so-called Fibonacci series, which we will study later. He also noted that during human growth these proportions are increasingly closer to the value of 0. He also observed that the navel divides the human body into two segments that have the golden ratio in relation to height.

We could cite numerous examples of the presence of proportion in our body. For example, on the following page, Figure 3.21 shows the proportions of a hand. As we can see, the proportion between the lengths of three of the phalanges of the hand again gives Φ.



$$\frac{A}{B}=\frac{B}{C}=\Phi$$

Figure 3.21. Golden proportions of the hand.

2.2. The golden ratio and vision

At the beginning of the chapter we showed the human preference for the golden rectangle over others of different proportions. We stated as a proven fact that these surfaces have a special appeal, and we showed different examples belonging to the fields of art and advertising. After all that has been said so far, a question arises: but why does this phenomenon occur?

In our relationship with our environment, we tend to prefer everything that we perceive clearly and that is not aggressive to our senses. We do not like flavours or smells that are too intense, but if they are very weak, we do not notice them. We are attracted to music at an audible level but without stridency, without excessive richness of high or low notes. During a conversation, we feel uncomfortable when our interlocutor speaks very quietly, but also when he shouts too much. If he is very far away, he forces us to raise our voices too much, which tires us, but if he is very close, he makes us uncomfortable because he occupies what is called "our vital space". To speak with someone, we usually place ourselves at an intermediate distance, so that the interlocutor is within our reach but without getting too close, looking at his face, but without staring at him. All these practices are the result of social customs, but also of our physiological constitution. Talking to someone without looking at them or standing too close to them causes us to miss certain aspects of the conversation that are only picked up unconsciously, such as the slight and subtle movements of the face and the rest of the body. We tend to position ourselves so that our interlocutor occupies the centre of our field of vision.

Bearing these considerations in mind, let's look at some anatomy.

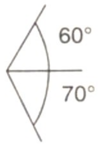


Figure 3.22. Vertical field of view of the human eye

Vertically, the eyes have an angular opening of 60° above and 70° below the horizontal of the eye.

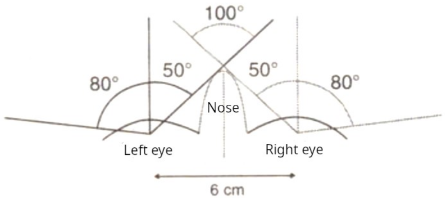
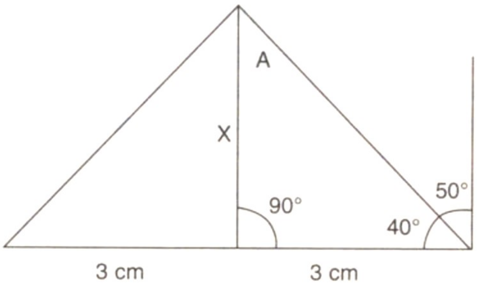


Figure 3.23. Horizontal field of vision of the human eye

Horizontally, in relation to the front of each eye, they have an external lateral opening of 80°, and an internal one (towards the nose) of 50°. Given these qualities, both eyes simultaneously dominate an area of 100°. The average distance between the eyes is 6 cm.

With these data, let's do some calculations.



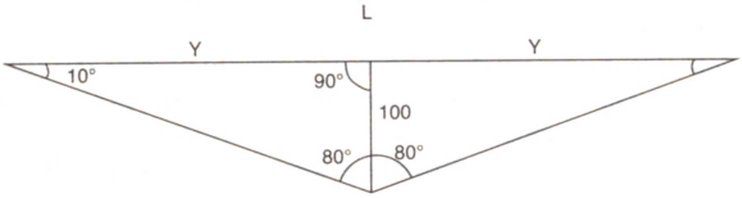
Suppose we want to find X and angle A in the triangle formed by the two eyes and the point of origin of the intersection of the visual field of the two eyes. We know that the sum of the angles of a triangle is 180°; with this we can establish:

$$A=180°-(90°+40°)=50°.$$

With the help of the sine theorem we obtain:

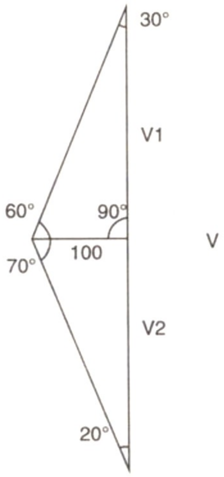
$$\frac{X}{\sin 40°}=\frac{3}{\sin 50°}\Rightarrow X=\frac{3-\sin 40°}{\sin 50°}\approx 2.5\,\text{cm}.$$

Now we will find out the proportions of a flat surface placed in front of our eyes. To minimize the effect of the separation between the eyes (6 cm) and to be able to consider them as a single point, we will place the surface at a distance sufficiently greater than 6 cm so that we can ignore them; we will take for example 1 meter (100 cm).



$$\frac{Y}{\sin 80°}=\frac{100\,\text{cm}}{\sin 10°}\Rightarrow Y\approx 567.1\,\text{cm}$$

$$L=2Y\Rightarrow L\approx 1{,}134.2\,\text{cm}$$



$$\frac{V1}{\sin 60°}=\frac{100}{\sin 30°}\Rightarrow V1\approx 173.2\,\text{cm}$$

$$\frac{V2}{\sin 70°}=\frac{100}{\sin 20°}\Rightarrow V2\approx 274.7\,\text{cm}$$

$$V=V1+V2\approx 173.2+274.7=447.9\,\text{cm}.$$

$$ratio=\frac{L}{V}=\frac{1{,}134.2}{447.9}\approx 2.532.$$

So far, these data do not present anything interesting for the topic at hand. The vertical angles are less than the external horizontal angle due to the visual limitations imposed by the bony area of the eye sockets, just as the internal horizontal angle is limited by the nose. This means that at the limit of these angles, vision is acceptably good, which does not occur with the external horizontal angle (80°), where at the limit vision is not good, since it enters into what is known as "peripheral vision" (we see approximate shapes, we cannot detail the exact shape and we only perceive the colour vaguely).

These differences in vision occur because the distribution of the optical sensors (cones and rods) is not uniform. For this reason, if instead of choosing 80° we choose a slightly smaller angle, we will have an area in which vision is more uniform, that is, we will perceive objects more correctly.

So, for example, we will take an angle of 75° instead of the previous 80°.

In this case we will have:

$$\frac{Y}{\sin 75°}=\frac{100\,\text{cm}}{\sin 15°}\Rightarrow Y\approx 373.2\,\text{cm}$$

$$L=2Y\Rightarrow L\approx 746.4\,\text{cm}$$

$$V\approx 447.9\,\text{cm}$$

$$ration=\frac{L}{V}\approx\frac{746.4}{447.9}\approx 1.66.$$

This measurement sounds familiar to us! The area where human vision is most comfortable has proportions very close to the golden ratio. A golden rectangle fits quite well into the comfortable vision zone of a human being.

But there is more. We know that the stereovision zone, that is, the zone of simultaneous control of both eyes, covers an angle of 100°. With this angle we will calculate the ratio of this vision zone.

Calculating the ratio for a central angle of 100° would be like calculating it, just as we have done so far, for a lateral angle of 50°.

$$\frac{Y}{\sin 50°}=\frac{100}{\sin 40°}\Rightarrow Y\approx 119.2\,\text{cm}$$

$$L=2Y\approx 238.3\,\text{cm}$$

$$V\approx 447.9\,\text{cm}$$

$$ratio=\frac{L}{V}\approx\frac{238.9}{447.9}\approx 0.53\,.$$

This result is not significant at all. However, if we calculate the inverse of 0.53, that is, if we look for the ratio between the longest and the shortest length, we will obtain:

$$ratio=\frac{L}{V}\approx\frac{447.9}{238.3}\approx 1.87$$

And this is a ratio that fits quite well with an area of golden proportions, although in this case the vertical component is greater than the horizontal one.

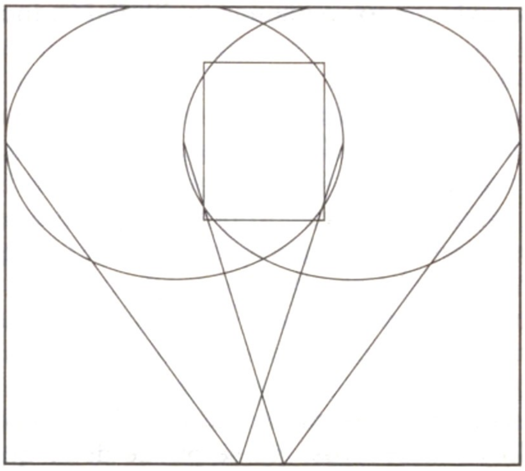
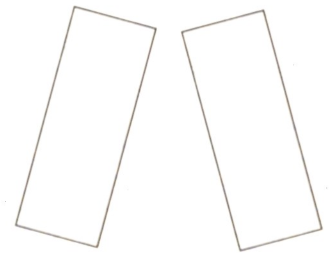


Figure 3.24. Vision zones of the eyes.

Taking this into account, we can perhaps explain a really curious fact. If we ask anyone to draw a rectangle for us, they will most likely draw it with the longest side in a horizontal position.

Few people will place the shorter side on the horizontal.

And even fewer people will draw it tilted.



However, the first one is just as rectangular as the second and the third ones. It cannot be said that the horizontal one is more golden than the vertical one, since apart from the 90° rotation, they are identical.

This experience allows us to check the influence of our anatomical properties (having our eyes side by side horizontally and the existence of peripheral vision) on our aesthetic preferences.

3. Fibonacci's contribution

3.1. Introduction

This section will allow us to provide a more formal overview of the golden number and the golden ratio in order to expand on the topic. The uninitiated reader can skip to the next chapter.

Leonardo Bonacci (1170-1240), an Italian mathematician known as Leonardo da Pisa or Leonardo Pisano because he was originally from Pisa, went down in history as Fibonacci, that is, the son of Bonacci. His father, a commercial representative of the city of Pisa in Algeria, was in contact with Arab culture and was especially interested in its mathematics. Fibonacci spread Arabic characters throughout the West and published the Liber abbaci (or Book about the Abacus), perhaps the most important work of medieval algebra, written in 1202 and of which only the 1228 version (the second version) survives. In this work he applies various algebraic problems to the resolution of commercial problems and explains the importance of the Hindu-Arabic numeral system.

We are going to highlight a problem that appears in this book (pages 123 and 124) that deals with the birth of rabbits. The subject had nothing significant until, at the beginning of the last century, it was the object of numerous studies that allowed to discover the properties of a special sequence of numbers: the one known as "Fibonacci series", although Kepler (in De Nive Sexangula) had previously related the Fibonacci sequence with the golden section and the growth of plants.

3.2. The Fibonacci series

The so-called "Fibonacci series" consists of an infinite succession of numbers whose first two elements are 1 and whose peculiarity is that each new term is obtained by adding the two previous ones:

1, **1**, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377,…

The general term of this sequence is:

$$\boxed{a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n\right]}$$

As can be seen, the expression for the golden ratio, $(1+\sqrt{5})/2$, is present in the Fibonacci sequence.

If we form another series in which each term is the quotient between a term of the Fibonacci series and the previous one, this new series tends towards the golden number (1.618...):

1/1; 2/1; 3/2; 5/3; 8/5; 8/13; 13/21; …
1; 2; 1.5; 1.667; 1.60; 1.625; 1.615; ...

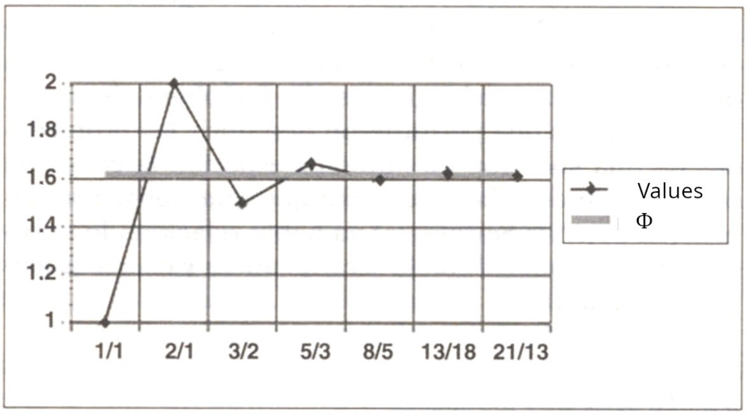


Figure 3.25. Convergence of the Fibonacci series

However, this does not end here, because if we form more series in which the condition that each term is the sum of the two previous ones is fulfilled, from them we can also obtain new series in which the quotient of each term by the previous one tends to the golden number.

Let's see some examples:

3;	**6**;	9;	15;	24;	39;	63;	102;	…
6/2;	9/6;	9/15;	24/15;	39/24;	63/39;	102/63;	…	
3;	1.5;	1.667;	1.6;	1.625;	1.615;	1.619;	…	
1;	**8**;	9;	17;	26;	43;	69;	112;	…
8/1;	9/8;	9/17;	26/17;	43/26;	69/43;	112/69;	…	
8;	1,125;	1,889;	1,529;	1,654;	1.605;	1.623;	…	
1.5;	**2.3**;	3.8;	6.1;	9.9;	16;	25.9;	41.9;	…
2.3/1.5;	3.8/2.3;	6.1/3.8;	9.9/6.1;	16/9.9;	25.9/16 ;	41.9/25.9 ;	…	
0.652;	1.652;	1.605;	1.623;	1.616;	1.619;	1.618;	…	

We can verify that for any pair of numbers we would obtain a sequence whose associated series of quotients tends towards the golden number (1.618…).

Other properties of Fibonacci numbers are:

a) The succession $f_1/f_2, f_2/f_4, f_3/f_5, f_4/f_6, \ldots, f_n/f_{n+2}$ converges to $\Phi^2 = 2.618\ldots$.

b) It can also be verified that the difference of squares of two Fibonacci numbers whose indices differ by two units is another Fibonacci number.

$$a_4=3 \text{ and } a_6=8$$
$$(a_6)^2-(a_4)^2=64-9=55=a_{10}$$

$$a_5=5 \text{ and } a_7=13$$
$$(a_7)^2-(a_5)^2=169-25=144=a_{12}$$

c) Another series can be formed in which each term is the sum of the three previous ones:

1, 1, 1, 3, 5, 9, 17, 31, 57,…

This series is called the "Tribonacci sequence."

d) To finish with the series, let's look at two more, related to the golden number, in which we can check the special mathematical qualities of this number.

Φ^n	A
1	1
Φ	Φ
Φ^2	$\Phi+1$
Φ^3	$2\Phi+1$
Φ^4	$3\Phi+2$
Φ^5	$5\Phi+3$
Φ^6	$8\Phi+5$
Φ^7	$13\Phi+8$
…	…

The first column is a series of successive powers of the golden ratio. In the second, as in the Fibonacci series, each term is obtained by adding the two previous ones; in this case, the first two are 1 and 0.

3.3. Rabbits and Fibonacci

Fibonacci, in his work Liber abbaci, raises a curious problem:

A pair of rabbits is placed in an enclosed yard to see how many offspring it produces in the course of a year, taking into account that each month, starting from the second month of life, a pair of rabbits gives rise to two more rabbits. In the first month the pair of rabbits have offspring, so we have two pairs. In the following month, the first pair have offspring again, so that in the second month there are three pairs. Of these, two pairs have offspring in the following month, so that in the third month two additional pairs of rabbits are born and the total number of pairs reaches five. In the fourth month, three of these five pairs have offspring and so the number of pairs reaches eight. Five of these pairs produce another five pairs, which, together with the eight existing pairs, give thirteen pairs in the fifth month.

Five of these pairs do not have children in this month, while the remaining eight have offspring, so that in the sixth month we will have 21 pairs. Adding to these the 13 pairs born in the seventh month, we have a total of 34 pairs. Adding to these the 21 pairs born in the eighth month, the total is 55 pairs. Adding to these the 34 pairs born in the ninth month, we have 89 pairs. Adding to these the 55 pairs born in the tenth month, we have a total of 144 pairs. Adding to these the 89 pairs born in the eleventh month, we arrive at a total of 233 pairs.

Finally, adding to the 233 pairs the 144 pairs born in the last month, we obtain a total of 377 pairs. This is the number of pairs produced by the first at the end of a year. Examining the table above, we can see that this result is arrived at by adding the first number to the second (1 to 2), the second to the third, the third to the fourth, the fourth to the fifth and so on, until adding the tenth and the eleventh (144 and 233), to obtain the total number of rabbit pairs (377).

Month	No. of pairs
Initial	1
1st	2
2nd	3
3rd	5
4th	8
5th	13
6th	21
7th	34
8th	55
9th	89
10th	144
11th	233
12th	377

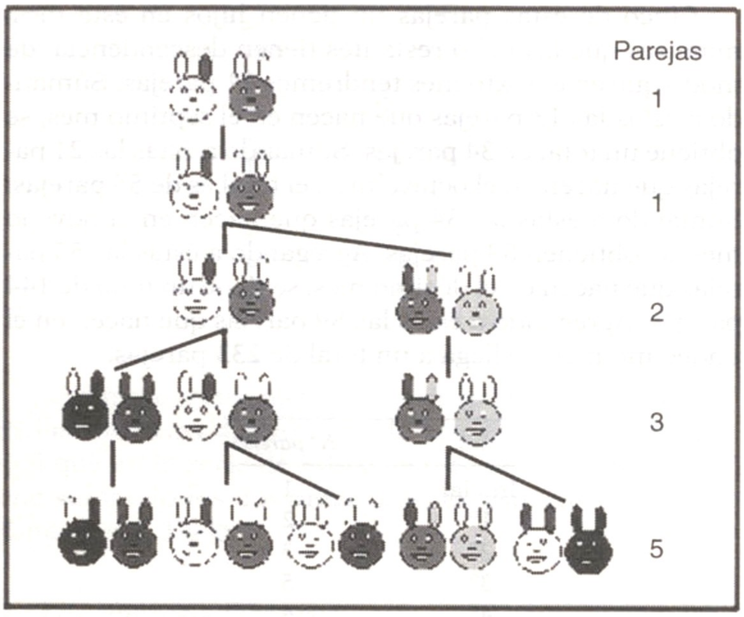


Figure 3.26. Fibonacci Rabbits

As can be seen, all the numbers thus obtained are terms of the Fibonacci series.

3.4. The Fibonacci series and the golden ratio in nature

Despite the artificial appearance of the Fibonacci series, we can find them in the nature around us. Look, for example, at the petals of flowers. Have you noticed that a large number of them have a number of petals equal to one of the numbers in the Fibonacci series?

The following photographs show the petals and leaves of a geranium and the sepals of a rose before it blooms. But we can think of many other plants, such as clovers, bougainvilleas, daisies and many others.

We can also find Fibonacci series in the helices that form the scales of the pine cones of conifers or tropical plants, or those that form the seeds of some plants such as sunflowers.

To see the helices properly, you have to look at the pine cones from the point where they join the branch. In this position you will clearly see that the scales are forming helices. They actually form two types of helices simultaneously, one clockwise and one anticlockwise. Interestingly, the number of blades of the helix in one direction and the other are different, but both belong to the Fibonacci series.

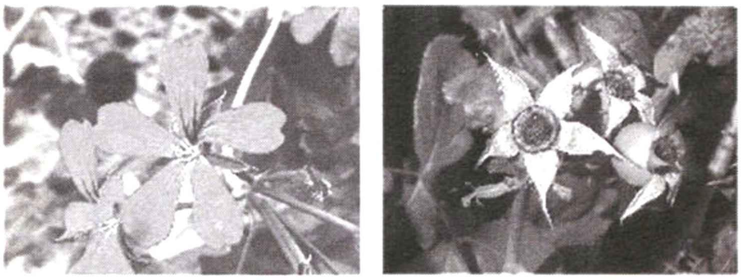


Figure 3.27. Geranium flower and rose sepals

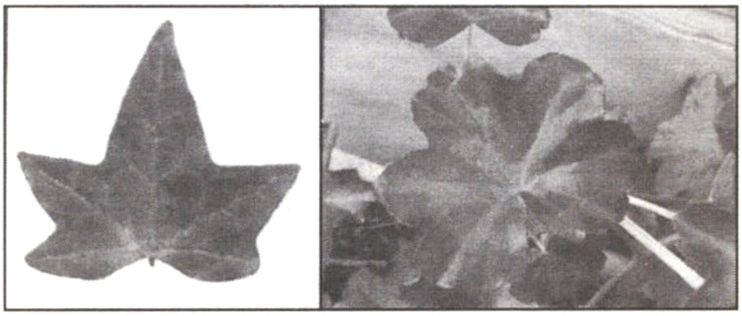


Figure 3.28. Pentalobe leaves

The image below shows a pineapple with eight curves clockwise and thirteen anticlockwise. Eight and thirteen are two consecutive numbers in the Fibonacci series.

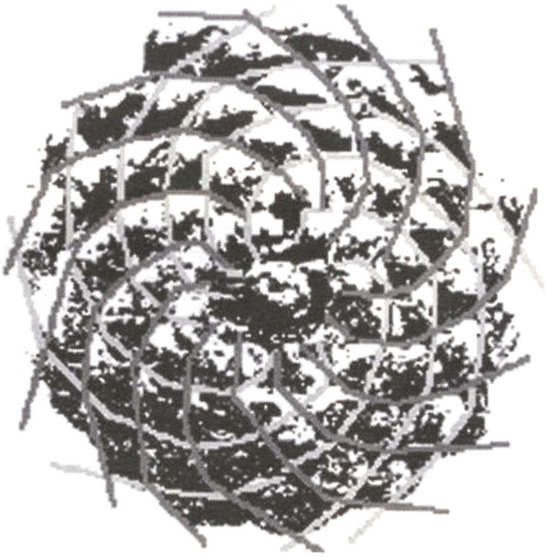


Figure 3.29. Arrangement of helices in a pineapple

And now let's look at a real case, like those we can see in any of our coniferous forests.

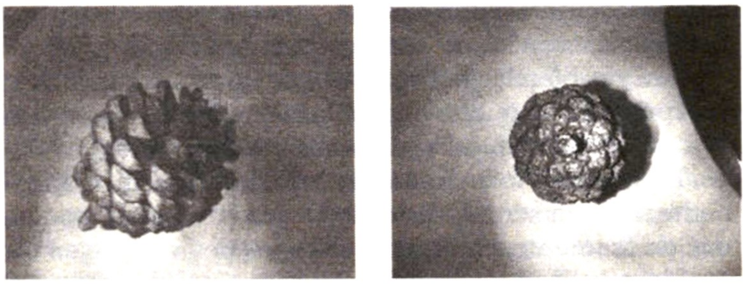


Figure 3.30. Pine cone

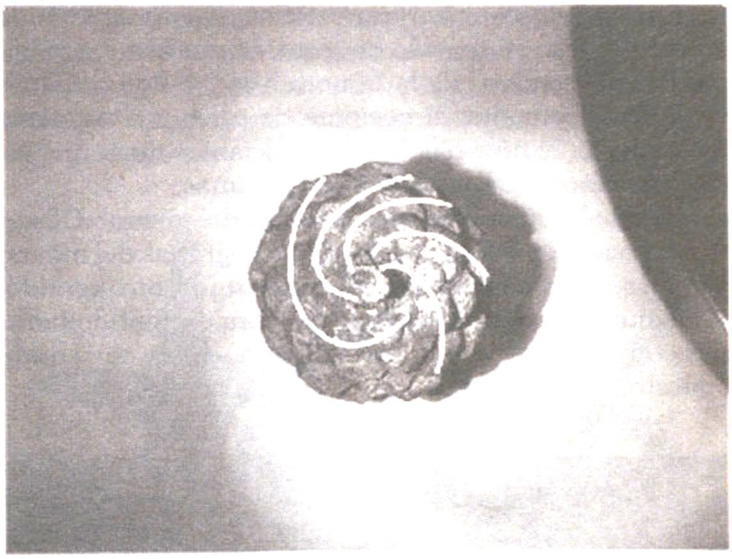


Figure 3.31. Clockwise Helix

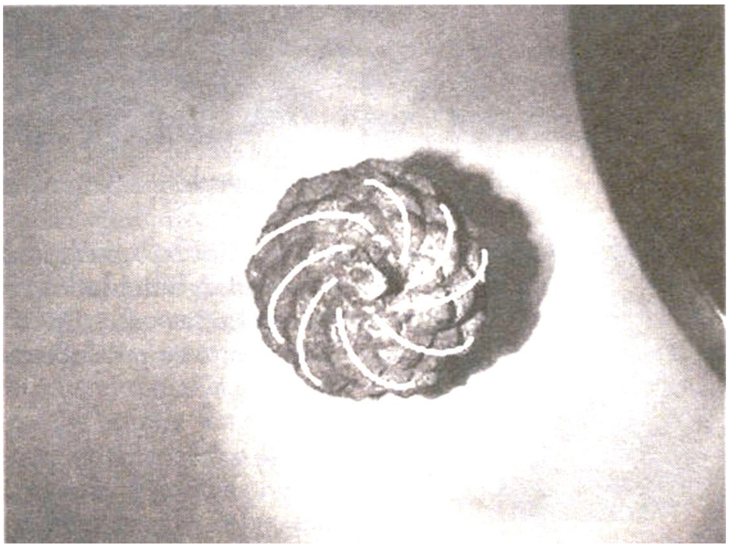


Figure 3.32. Anticlockwise helix

In these cases we find five clockwise curves and eight anticlockwise curves, again two consecutive numbers from the famous series. And as we have seen previously, the quotient of these two numbers tends towards the golden number; we could say that these pineapples hide gold between their scales.

The same thing happens with sunflower seeds. Look at the images below. In two photographs of the same sunflower, the spirals [flat helices] have been indicated in one direction or another: on the left, we see the spirals that rotates clockwise, with 21 curves, and on the right the one that rotates anticlockwise, with 34 curves.

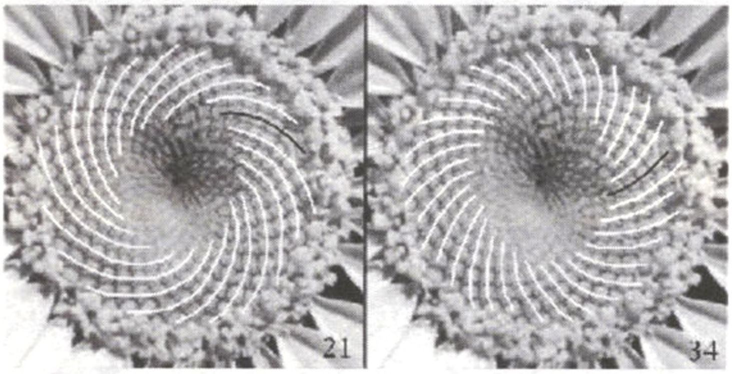


Figure 3.33. Helices of a sunflower

And finally, another curiosity. Phyllotaxy is the position of the buds of plants along the stem. In some plants the buds are placed every 137.5°, that is, from one bud to the next there is a turn of 137.5°, or what is the same:

$$360^\circ\left(1-\frac{1}{\Phi}\right)=137.5^\circ$$

As we can see, the golden ratio and the Fibonacci series are still closely related.

So far we have analysed the plant world, but we can also find the presence of the Fibonacci series in the animal world. Just as an example, let us think of the starfish, with its five arms in a pentagonal arrangement.

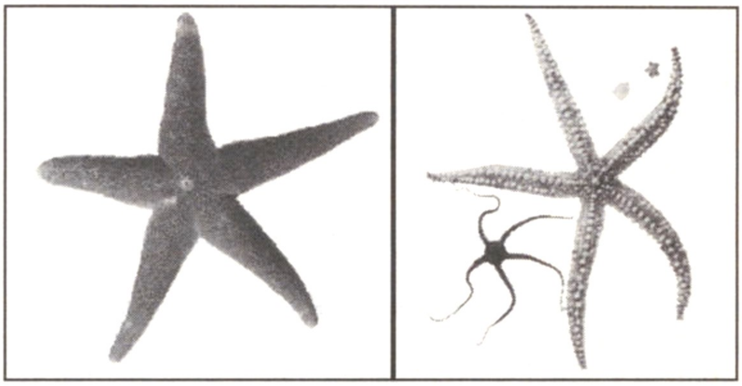
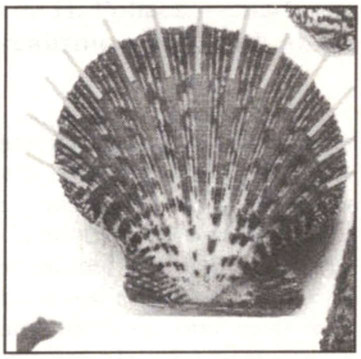


Figure 3.34. Starfish

Without leaving the sea we can find more examples, such as the main grooves on the shells of some molluscs. In the following image we have the shell of a scallop, a marine mollusc, with its thirteen main grooves marked.

On any beach we can find a multitude of shells abandoned on the shore by the waves. If we look closely, we will see that, among all the types we can find, those with the most marked grooves have a number of them that coincides with a term in the Fibonacci series.



Scallop shell

IV. MATHEMATICS AND SOCIETY

Scientists strive to make the impossible possible, politicians to make the possible impossible.

Bertrand Russell (unverified)

1. Mathematics, a social fact

Throughout the book I have stressed the need to put an end to the boring and useless image that dominates mathematics and its traditional teaching, and I have shown its influence on everyday life through some examples. These examples demonstrate that mathematics education should be broader. It should be a true education in the humanities, in which students are not limited to learning the specific contents of the subject, but are also aware of the role that mathematics plays in our culture and society. As an illustration, we will recall some words by Henry O. Pollak —quoted in «Apología de la utilidad y del realismo» [“Apology for Utility and Realism”] by Claudi Alsina (1996)—:

> Traditionally, the Mathematics of everyday life has been the Mathematics of primary school. Mathematics for exercising intelligent citizenship should basically be the Mathematics of secondary school. Mathematics for the profession should be the one taught at the university stage (if the profession requires studies at this level). Mathematics as part of the integral human culture has not been assigned to any educational level.

[The English translation from the Spanish translation is untraceable but the following is from Henry Pollak’s paper presented that the Third International Conference on the Teaching of Mathematical Modelling and Applications, Kassel University, 8-11 Sept. 1987:

> Mathematics for practical everyday life is traditionally the responsibility of the elementary school. Mathematics for intelligent citizenship — and we shall discuss this considerably more in what follows — should be the responsibility of the secondary school. At least, that is my opinion. Mathematics for your profession is the responsibility of tertiary education, at least for those professions which require advanced education; otherwise it is also a responsibility of the secondary level. To teach mathematics as a part of overall human culture is, as far as I can see, nobody’s specific assignment.

{Applications and modeling in learning and teaching mathematics, edited by W Blum et al. 1989 https://archive.org/details/applicationsmode0000unse }]

Mathematics has made realistic painting possible and has made it possible to understand musical sounds; the analysis of these sounds has been fundamental for the construction of telephones, radios and other sound recording and reproduction devices. Mathematics appears in research in biology, medicine and economics, and is essential in today's society.

Knowledge is a whole and mathematics is a subset of this whole. Teaching mathematics as if it were isolated is a distortion of knowledge. Each subject represents an approximation to knowledge and any pedagogically useful contribution must be welcomed. In this way, it would be appropriate to teach mathematics by going beyond mathematics itself: considering its relations with other stages – social and human interests – and seeking its harmony with the main currents of human and technological thought. This new attitude would provide motivation to students, create new applications and open new avenues of debate.

Mathematics is not a set of isolated knowledge, but rather it responds to specific purposes and objectives. As teaching professionals, we must demonstrate its usefulness beyond the purely mathematical field. It is desirable that students and citizens - who are the users of mathematics - know the daily and professional use of mathematics. In fact, if we show them the usefulness of mathematics, students will feel more motivated to study it, which will have a positive impact on their career.

Historically, the motivation behind the development of mathematics has been the solution of real-world problems—mostly physical ones. Newton studied the motion of the Moon [and the subsequent understanding of that motion was made] to help sailors determine their position at sea. Euler studied ship design to create navigational charts. Descartes designed lenses to improve the telescope and microscope. Gauss worked to perfect the electric telegraph and the measurement of magnetism. Mathematics is a means to an end! Concepts and reasoning are used to obtain answers to real-world problems.

Citizens, and especially students, live in a real world, and they are certainly more attracted by everyday phenomena than by the abstraction of the professional mathematician. Let us recall Plutarch's beautiful statement:[1] "The mind is not a vessel to be filled, but a fire to be kindled." [https://quoteinvestigator.com/2013/03/28/mind-fire/]

Negative numbers are not simply the inverse of positive numbers in addition, they are also the degrees that a thermometer marks below zero; the ellipse is not simply a geometric place, but is also the trajectory of a planet or comet. Functions are more than ordered pairs, they are also relationships between the laws of the universe and those of society. In short, mathematical concepts are the fruit of real phenomena. Hiding the meaning of these concepts is like keeping the fishbone and despising its flesh.

Society is becoming increasingly formalized and mathematized as a result of large-scale economic and technological changes, in which mathematics plays a very important role. We could say that mathematics is the driving force of development and of technological and social changes.

This rapid progress requires a higher level of professionalism. Today's students must be prepared to be citizens of the future, with the criteria and skills necessary to continue learning. We cannot make the mistake of believing that training ends when students finish their degree. The technical knowledge acquired twenty years ago is no longer valid; social and work dynamics cause constant retraining. And this retraining must primarily affect teachers. If students twenty years ago studied the behaviour of steam engines, today's students study the behaviour of diesel engines; but we need to be more up-to-date. Educators need to make a firm commitment to new technologies.

Furthermore, the teacher must be able to justify the presence of mathematics in education at all times. It is obvious that, if there were no justification, it would be omitted. But it is a matter of convincing the student of the need to learn mathematics because this is not exactly the most well-received subject. We can point out some arguments to justify the presence of mathematics in educational plans:

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[1] ‘quoted by Morris Kline (1986)’ no reference found.

1. Mathematics plays a very important role in current technical development.
2. Mathematics is part of human culture.
3. Mathematics can offer content of curricular interest at a formative and informative level.
4. Mathematics enriches creativity and critical thinking.

The above arguments are supported by various authors, such as Claudi Alsina, John Perry or Morris Kline.

In 1919, the Mathematical Association noted:[2]

> The education we offer a student should prepare him to be a citizen in the broadest sense of the word. His education should enable him not only to apply mathematics to practical matters, but also to understand the great problems of the world, the solution of which depends on mathematics and science.

Zoltan Dienes stated in 1978:[3]

> The main goal of mathematics must be the development of certain patterns of thought, certain strategies, that people can develop when faced with new situations, which they have not previously encountered.

Let us note that in the above quotes we find utilitarian reasons for mathematics and its teaching, professional indications and needs, development of formative capacities and development of personality and one's own aptitudes.

It would be unfair not to quote Mogens Niss, former secretary of the International Committee on Mathematics Education (ICME), who stated in 1992:[4]

> We must carry out an analytical reconstruction of the function of mathematics in the world—that is: nature, society and culture—taking into account that this role varies according to time and place.

Niss expresses the teaching of mathematics algebraically in the following way:

> The teaching of mathematics depends on mathematics in society, mathematics in culture, mathematics in politics and economics, and on the individual and his or her cultural and ideological values.

Mathematics teaching must therefore contribute to stimulating the intelligence and curiosity of all members of society. More specifically, mathematics teaching must take place everywhere, in order to create a perspective of the whole in general, that is, of the essential forces that exist side by side in the development of nature, society and the life of people.

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[2] Unverified.
[3] Unverified.
[4] Unverified.

2. Innovation and new technologies

> To raise new questions, new possibilities, to regard old problems from a new angle, requires creative imagination and marks real advance in science.
>
> Albert Einstein
> *The Evolution of Physics*, with Leopold Infeld (1938).

The broad impact of technology and the rapid development of different areas of knowledge require that academic curricula be constantly reviewed to verify internal coherence and the effectiveness of teaching.
It is relevant, within this review, to study a model of mathematics teaching that incorporates new trends in mathematics education, the aim of which is to insert mathematical science into society as an intellectual tool of habitual use among citizens. In the case of compulsory and subsequent education, this new vision is fundamental, since it would add to the academic profile of mathematics education a critical, innovative and creative aspect that would contribute to achieving an optimal professional profile.

This new way of teaching mathematics is based on the hypothesis that the brain is not a massive consumer of knowledge, but rather constructs it through a process that includes selection, interpretation and inference.

Another aspect to consider is the spread of new technologies, which are currently applied to a wide range of daily activities. The computer has become an essential tool on the desk of any office, in a doctor's or lawyer's office, and at the checkout in a supermarket. Audiovisual media (television, video, Internet) are witnesses to the intimacy of many families, as if they were another member.

Education cannot be an exception, it must not be excluded from the use of these media. One of the missions of education must be to enable citizens to understand the culture of today's world. In this sense, new technologies must serve to improve the quality of teaching and to ensure that, as far as possible, methods close to the professional environment are used in it, thus achieving a closer approximation to reality. Therefore, it is necessary to consider that for teaching we do not only have books and the traditional blackboard, but we also have—to a greater or lesser degree—other valid and necessary tools for improving the quality of teaching: overhead projector, videos, computers, audio equipment.

The use of audiovisual and computer media in the classroom requires a predisposition on the part of the teacher. Often, ignorance, lack of resources and lack of training make the implementation of new technologies difficult.

It is clear that the simple use of a transparency already represents a small methodological change with respect to the book-blackboard-teacher triad. A transparency allows students-interlocutors to present summaries of the most important points of the topic and to show diagrams and tables with greater clarity, quality and sharpness than traditional chalk. Audiovisual media favour the presentation of images, animated or not, of the topics discussed, help to identify the topics with reality, stimulating knowledge and increase the quality of the drawings and representations in relation to what we can achieve by drawing on the blackboard or on paper. And if we combine image and sound, we will encourage greater attention on the part of the students. Videos and recent DVDs can be used to introduce topics or to develop certain concepts.

However, we must be aware of the limitations of new technologies and act with caution. It is important to be clear that new technologies are not cosmetics: it is not about continuing as always and showing on video the same things that were previously drawn on the blackboard with chalk. An outdated curriculum continues to be out of place whether it is supported by the traditional blackboard or disguised by a video.

Computers, and specifically the computer, provide a quick and convenient way of accessing and representing information. The computer allows students to spend more time understanding and analysing results than mechanical and manual methods. In addition, possible calculation errors are minimised. The computer allows experimentation with real data, which is very difficult and complicated to work with manually. The typical argument—common in various educational stages —“How can we explain this topic in fewer hours?” is no longer valid; it is necessary to reorient the teaching of mathematics by placing more emphasis on intuition, modelling and new technologies, without dwelling so much on formal deductive processes. It is also necessary to avoid the abstraction that is characteristic of mathematics for mathematicians.

I believe that the use of new technologies encourages group work, debate and discussion among students—an aspect that undoubtedly enriches learning—as well as the acquisition of skills for cooperative work, while helping to produce written reports of the activities viewed on the computer or in the video.

3. Recommendations for improving teaching quality

3.1. The use of computers

1. It makes it possible to solve complex problems and use real data from the earliest years of schooling.

2. It makes it possible to overcome routine tasks, in favor of a deeper attention to problem solving and the more conceptual aspects of applications.

3. Some problems that are theoretically inaccessible, due to their complexity or the requirement of mathematical knowledge, can be studied through numerical or graphical simulation with the help of a computer. This is possible thanks to the existence of computer programs that are available to students, among which the most popular are Mathematica, Derive, Matlab and Maple. In addition, on the market today we can find calculators that perform symbolic calculations and that can serve as a complement in the classroom; among them, the TI92 from Texas Instruments stands out, which incorporates “Cabri Geometry” and an adapted version of Derive. The use of these tools allows calculations to be carried out that until recently required a lot of time and in which it was not always possible to work with real data. These aspects entail a change in the methodological content, while favouring the replacement of obsolete algorithms and a greater dedication to the meaning of the concepts and their immediate applications.

3.2. In the field of content

The content needs to be revised. The mathematics syllabuses taught in schools are practically the same regardless of the curricular orientation (economics, biology). Some of them have become outdated with respect to the needs of everyday life and future professional life. Textbooks for users of mathematics must be different from mathematics books for professional mathematicians.

3.3. In the methodological field

Not only the content must be redefined, but also the methodologies.

1. It is necessary to take into account the impact of mathematics on the student's professional future: it is advisable to include modelling practices as a heuristic process of knowledge construction with the aim of training critical professionals. The teacher will present a real-world situation, encouraging students to build the corresponding mathematical model.

2. It is necessary to promote group work. In professional life, citizens work in collaboration with other colleagues, sharing experiences and advancing in the development of projects. We must avoid individualism and therefore think of new ways of assessing. In a society where group work and the use of new technologies are increasingly valued, we find—in contrast to social demands—that schools still require students to take maths tests without being able to use calculators or notes, ask questions or debate with classmates. One recommendation on how to assess would be to publicly defend a project carried out in a group so that classmates participate in a debate resulting from the presentation. It is necessary to assess what students know, not what they do not know.

3. It is necessary to create a mathematics workshop that includes practices with videos, audio and computers connected to the Internet.

3.4. The teacher's attitude

1. It is necessary to insist on the social importance of the work of the mathematics educator. The role of the teacher should not be that of a simple speaker and evaluator, but rather his work should consist fundamentally of providing guidance.

2. We must be the first to demand the highest quality and scientific seriousness in our work.

3. It is necessary to reorient the initial and ongoing training of mathematics educators, seeking a balance between the mathematical and psycho-pedagogical components.

4. It is necessary to insist on cooperative work. The educational dimension is also social and, therefore, it must be practised socially, in work groups, seminars and departments that promote joint actions and contribute to creating a state of opinion on the problems of teaching and learning mathematics.

4. Pedro Puig Adam's legacy

Pedro Puig Adam dedicated his entire life to teaching and didactics of mathematics. The pedagogical renewal that his ideas and initiatives represented has had a great impact at an international level. I would like his wise advice—set out in the «Decálogo de la didáctica de las matemáticas en la enseñanza media»[5]—which has not lost its relevance and is part of the debate and reflection on education, to be reflected in the present text.

Below I will detail Pedro Puig Adam's educational legacy:

1. Do not adopt a rigid teaching method, but rather adapt it to each student, observing them constantly.
2. Do not forget the specific origin of the mathematics or the historical processes of its evolution.
3. Present mathematics in unity with natural and social life.
4. Carefully graduate the levels of abstraction.
5. Teach by guiding the student's creativity and invention.
6. Stimulate said activity by awakening a direct and functional interest towards the objective of knowledge.
7. Promote, as much as possible, self-correction.
8. Gain some mastery of the solutions before automating them.
9. Ensure that the student's expression is a faithful translation of his or her thoughts.
10. Ensure that students always achieve successes to avoid discouragement.

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[5] DECÁLOGO DE LA DIDÁCTICA MATEMÁTICA MEDIA (1955) by Pedro Puig Adam

EPILOGUE

The aim I pursued in writing this book was to offer a different vision of mathematics and its teaching, so that the word "mathematics" does not provoke fear and a desire to run away among citizens. We could characterize this vision in the following points:

1. *Training aspect*: stimulate interest in discovery and creativity and acquire confidence in the subjects' abilities and resources.

2. *Critical competence*: In a society increasingly influenced by mathematics, through its applications and models, it is necessary to develop critical competence among students that allows them to integrate into the world of work and society in a more active and participatory way. Critical competence must be understood as an ability to recognize, understand, analyse and validate the use of mathematics in a real-life context.

3. *Utilitarian aspect*: the ability to apply mathematical knowledge to situations in the social world does not come from abstract training. As experience shows, it is related to a more practical prior education. To achieve this objective, it is necessary to carry out modelling practices with students, so that they acquire valid and useful tools for their future professional careers.

4. *Integrative vision of mathematics*: it is necessary to present mathematics as a tool linked to other branches of science and as a cultural and social activity.

5. *Psychological argument*: the incorporation of mathematical applications in the curriculum can make mathematical concepts more important for students. The effectiveness of mathematics and its constant presence in modern society lead to the suggestion of using mathematical thinking to model practical situations. The ability to use a mathematical concept encompasses more than just simple knowledge of this concept. Knowing how to perform a calculation is no guarantee that one knows how to decide in which situations it is necessary to perform it or how to use the result. It is necessary to know how to distinguish whether the student is a mere spectator of procedures, who memorizes the techniques mechanically, or whether he understands what he is doing, understands the concepts and his interpretation really observes their usefulness.

More points similar to the above could be enumerated; the reader can complete the list.

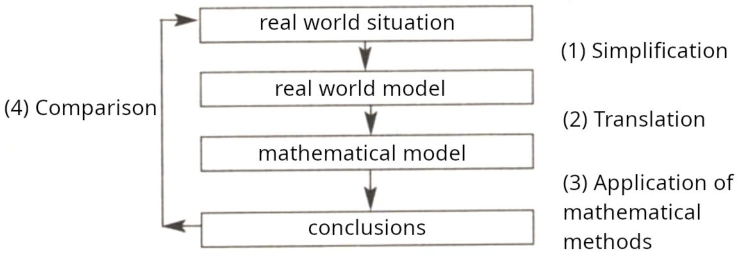
With the aspects and examples shown in this text, I intend to offer a different vision of mathematics, which I hope will be useful.

Ultimately, the optimal solution is to take advantage of educators who are inactive—perhaps due to legal requirements—and who have a long teaching career. This, together with new technologies, can ensure that today's children become citizens of tomorrow, prepared to face the challenges of the 21st century. Ultimately, teachers shape the professionals of the future. The profession of education represents a very noble and important role.

A methodological proposal

Personally, I believe that any essay on mathematics education should include a methodological proposal. For this reason, I have considered it appropriate to offer a methodological approach—not necessarily shared by the educational population—that I consider efficient and that has been tested at the Universitat Politécnica de Catalunya, among other educational centres.
The proposed method for presenting mathematical content in classrooms is based —as I point out in the introduction— on what is called “mathematical modelling as a teaching-learning tool”. Mathematical modelling consists —briefly— in formulating an everyday problem in mathematical terms —what I call “building a model”—, solving it, if possible, and interpreting the results in terms of the problem and the situation posed.

To illustrate what the modelling process is as a methodological proposal, I attach the following flowchart:



MODELLING PROCESS OUTLINE

The methodological proposal is focused on the following points:

1. Presentation of a simplified real-world situation.
2. Translation of the situation into mathematical terminology and obtaining the model.
3. Work on the model and problem solving.
4. Presentation of the solution in non-mathematical terms.

The proposal is therefore focused on teaching innovation and the viability of mathematical modelling. The viability of this methodology is developed in my work «Per un nou ensenyament de les matemátiques» [For a new teaching of mathematics], from 1999; here we can find recent experiences that support mathematical modelling and proven examples of its quality.

In this work I have included situations and reflections that show the human and fun side of mathematics and its presence in everyday situations from a different point of view than the traditional one. This fact involves a contribution of resources that could be included in teaching.

This need for change is a natural consequence of the speed of technological and lifestyle changes. We need to prepare our students for the world of today and tomorrow, and this forces us to think about the most effective way of teaching and learning.

In fact, mathematical modelling is becoming increasingly useful. The importance of mathematics lies in its application to specific or particular problems. The growing importance of mathematics teaching in society and culture is reflected in the growing tendency to solve everyday and professional problems mathematically. Mogen Niss defined modelling in 1991 as “the art of applying mathematics to real life”.

In fact, mathematical modelling could be characterised as an innovative and efficient teaching tool and as a transmission belt that provides the acquisition of knowledge and helps to strengthen ties between mathematics and reality.

At this point I believe that the reader will be able to answer the question posed by Albert Einstein (1938):

> How can we explain that mathematics, a product of the human mind, independent of experience, fits so well with the objects and elements of reality?
>
> [The quote “How can it be that mathematics, being after all a product of human thought which is independent of experience, is so admirably appropriate to the objects of reality?” by Albert Einstein can be found in his lecture "Geometry and Experience". This lecture was delivered at the Prussian Academy of Science in Berlin on January 27, 1921, and later published in various collections of his works.]

Finally, and as a reflection addressed to the entire educational community, regardless of the subject they teach, I would like to express some beautiful and moving words taken from the text “To a Very Special Teacher”, by Pam Brown. They reflect the impact that education should have on our students and are a good example of what we would like them to tell us:

> Thank you for making learning not a job but a joy. … Thank you for being someone I can always trust – and turn to when life gets difficult. … You never make us feel that you are just stuffing us with facts – you go with us on voyages of discovery. … You have taught us to have adventures in our heads – to search and discover, to live with amazement. … A teacher takes the everyday things of life, the things a child knows, and fashions them into stepping stones of knowledge and growth. … Thank you for never being sarcastic – the thing that baffles and bewilders a child. … Whatever chaos reigned outside – we know we’d find order, justice and a chance to learn inside your classroom. And tolerance. And laughter. And excitement. … You put beauty in my hand. … You, as teacher, gave me words, images, ideas, from which to build my life. Whatever I build you helped lay the foundations.
> [To a very special teacher : Brown, Pam]

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Vilanova i la Geltrú, November 2001

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